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Sponsor Contact Person (s):

Administrative
Mr. Wilbur W. Bolton, Jr.
Grants Officer
National Science Foundation
Washington, D. C. 20550

Technical Matters
Mr. Elias Schutzman
Electrical Sciences & Analysis Program
Division of Engineering
National Science Foundation
Washington, D. C. 20550

Assigned to: School of Electrical Engineering

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RESEARCH PROJECT TERMINATION

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ON AN ALGEBRAIC THEORY OF SYSTEMS DEFINED BY CONVOLUTION OPERATORS*

E. W. Kamen

School of Electrical Engineering

Georgia Institute of Technology

Atlanta, Georgia

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ABSTRACT

For a large class of linear continuous-time systems including delay-differential systems, an algebraic theory is presented in terms of Noetherian operator rings generated from a finite number of elements belonging to a convolution algebra of distributions. The external behavior of these systems is given by a finite set of input/output (convolution) operator equations which are solved by constructing an operational transfer function matrix defined over the quotient field of the operator ring. After the formulation of an internal representation consisting of a finite set of scalar operational-differential equations, the problem of realizing an operational transfer function matrix by such an internal description is considered. Results on the existence and construction of realizations are given.

I. Introduction

The practicality of existing theories on finite-dimensional time-invariant systems is mainly a result of "finiteness" in the structural aspects of the mathematical representation; for example, the finite degree of polynomials in the transfer function matrix or the finite size of matrices in the classical state space description. Unfortunately, these nice characteristics are lost when the theory is extended to include infinite-dimensional systems. As a result, in the infinite-dimensional case the usual transform and state space techniques do not yield practical computational procedures unless an approximation theory is implemented.

However, in many cases the infinite elements (or "devices") within a system play an integral role in system behavior such that approximations (by lumped elements, say) can not be made without losing the direct relationships between system properties and the characteristics of the infinite elements. A very common example is a continuous-time system containing ideal time delays with important system properties depending on the magnitudes of the time delays.

In the study of infinite-dimensional systems there is a need for an algebraic theory whose structural properties are "finite" so that computations within this framework can be performed. Algebraic techniques have been applied to various special situations, but a sufficiently general algebraic framework is still lacking even in the time-invariant case. The purpose of this paper is to establish the foundations of such a theory for a large class of infinite-dimensional time-invariant continuous-time systems including delay-differential systems.

The basic idea of the approach given here is to construct a one-to-one correspondence between the "primitive" elements of a system and the generators of the mathematical representation of the system. More precisely, given a class of systems consisting of an interconnection of elements having impulse responses $\theta_1, \theta_2, \ldots, \theta_q$ belonging to a convolution algebra V of distributions, the systems in this class are represented by (convolution) operators belonging to the subring of V generated by $\theta_1, \ldots, \theta_q$. The study of the input/output representation in terms of these operators is given in Sections 2, 3, and 4.

The finiteness of this algebraic framework is primarily a result of the fact that the ring of operators is a Noetherian domain which means that every ideal is finitely generated. Furthermore, in many cases the elements $\theta_1, \ldots, \theta_q$ are algebraically independent (as defined in Section 2) with the result that the operator ring is isomorphic to the polynomial ring in q symbols. In these cases the rich theory of polynomials in several symbols can be utilized to study the given class of systems. For example, the description of operational-differential systems in terms of polynomials in several symbols makes it possible to compute internal representations from an operational transfer function matrix. This is pursued in Sections 5 and 6. An example of the realization procedure is given in Section 7.

2. Input/Output Description

Let R denote the field of real numbers and let V denote the linear space of R-valued Schwartz distributions (generalized functions) defined on R with supports bounded on the left. As proved by Schwartz [1], with addition (+) and convolution (*) V is also a commutative ring with no divisors of zero. (In other words, V is an integral domain.) Further, the linear structure on V is compatible with the ring structure in that V is a convolution algebra over R. The multiplicative identity of V is δ_o, the Dirac distribution (unit impulse) concentrated at the origin. Since $\delta_o \in V$, R can be viewed as a subring of V under the embedding $R \to V : a \mapsto a\delta_o$.

Letting U denote a fixed linear subspace of V, we shall be concerned mainly with systems consisting of an interconnection of devices having inputs belonging to U, outputs belonging to V, and impulse responses belonging to V. In particular, given a single-input single-output device, the input/output behavior is specified by the linear operator $U \to V : u \mapsto \theta * u$ where $\theta \in V$ is the impulse response of the device. Common examples of such devices are integrators (θ = Heaviside function) and scalors for which $\theta = a\delta_o$, $a \in R$.

As is well known, if input/output measurements are taken at fixed points in one-dimensional Euclidean space, operators of the form $u \mapsto \theta * u$, $\theta \in V$, can also represent devices that are distributed in an axial direction. For example, $\theta = \delta_a$, $a > 0$, can be viewed as the impulse response of an LC transmission line (or ideal delay line) with time delay a. Elements of V are also utilized to specify lossy and dispersive delay lines of various types.

Generalizing, it is true that many devices of practical interest can be specified by an input/output operator $U \to V: u \mapsto \theta * u$ with $\theta \in V$. However, V may be unnecessarily "large" for some applications, in which case a theory corresponding to that given below could be constructed in terms of a proper subalgebra of V or some other convolution algebra. Also we could consider convolution algebras defined over the field of complex numbers.

The systems under consideration here are described in terms of (convolution) operator equations that are generated in the following manner. Given a finite list of fixed elements $\theta_1, \theta_2, \ldots, \theta_q$ belonging to V, let $R[\theta_1, \ldots, \theta_q]$ denote the smallest subring of V containing $\theta_1, \ldots, \theta_q$ and R (viewed as a subring of V). An element $\alpha(\theta_1, \ldots, \theta_q)$ in $R[\theta_1, \ldots, \theta_q]$ can be written as a finite sum

$$\alpha(\theta_1, \ldots \theta_q) = \sum_{j_1, \ldots, j_q} a_{j_1, \ldots, j_q} \theta_1^{j_1} * \theta_2^{j_2} * \ldots * \theta_q^{j_q}$$

where the j_i are non-negative integers, $a_{j_1, \ldots, j_q} \in R$, $\theta_i^j = j^{\text{th}}$ fold convolution of θ_i, and $\theta_i^o = \delta_o$. The ring $R[\theta_1, \ldots, \theta_q]$ is an integral domain since it is a subring of the integral domain V.

Any fixed $\alpha(\theta) \in R[\theta_1, \ldots, \theta_q]$ (θ denotes the list $\theta_1, \ldots, \theta_q$) defines a linear operator $V \to V: v \mapsto \alpha(\theta) * v$. With the usual addition and composition, the set of all operators on V of the form $v \mapsto \alpha(\theta) * v$, $\alpha(\theta) \in R[\theta_1, \ldots, \theta_q]$, is a ring which is isomorphic to $R[\theta_1, \ldots, \theta_q]$. For this reason, we shall sometimes refer to $R[\theta_1, \ldots, \theta_q]$ as a ring of operators.

For any fixed operator ring $R[\theta_1,\ldots,\theta_q]$, we consider the class of m-input terminal k-output terminal systems whose external description is given by the following finite set of (convolution) operator equations

$$(1) \qquad \sum_{j=1}^{k} \alpha_{ij}(\theta)*y_j = \sum_{j=1}^{m} \beta_{ij}(\theta)*u_j, \quad i = 1,2,\ldots,k$$

where $\alpha_{ij}(\theta)$, $\beta_{ij}(\theta) \in R[\theta_1,\ldots,\theta_q]$, the $y_j \in V$ are the outputs, and the $u_j \in U$ are the inputs, U = fixed linear subspace of V.

If $q = 1$ and $\theta = \delta_o^{(1)}$ = first derivative of δ_o, then since $\left(\delta_o^{(1)}\right)^n * v = \delta_o^{(n)} * v = n^{\text{th}}$ derivative of $v \in V$,

(1) is a set of ordinary linear constant differential equations which is often taken as the input/output representation of a finite-dimensional time-invariant system. If $q = 1$ and θ is any fixed element of V, then (1) could be the external representation of a system consisting of an interconnection of a finite number of adders, scalors, and devices having impulse response θ (or θ^{-1} if θ is invertible in V). More generally, if $\theta = \theta_1,\ldots,\theta_q$, then (1) could represent a system consisting of an interconnection of adders, scalors, and finite combinations of devices having impulse responses $\theta_1,\ldots,\theta_q$ (or θ_i^{-1} if θ_i is invertible in V). This latter case includes a large class of infinite-dimensional systems which are defined in terms of the following notions.

A device with impulse response $\theta \in V$ is said to be finite (or lumped) if there exist elements $\alpha, \beta \in R[p]$, $p = \delta_o^{(1)}$, such that $\beta*\theta = \alpha$. A device is infinite if it is not finite. Via standard constructions in realization theory, it can be shown that a device admits a finite-dimensional state space representation if and only if it is finite, hence the motivation for the term finite. Integrators and scalors are two common examples of finite devices. Examples of infinite devices are ideal delay lines and dispersive delay lines.

If we then let $\theta = p, \theta_1, \ldots, \theta_r$ where $p = \delta_o^{(1)}$, the set of operator equations (1) could represent a system consisting of an interconnection of finite devices and finite combinations of infinite devices having impulse responses $\theta_1, \ldots, \theta_r$ $\left(\text{or } \theta_i^{-1}\right)$. These systems will be referred to as operational-differential systems. Common examples are delay-differential systems in which the infinite devices are ideal delay lines.

An obvious but important point is that the properties of a system specified by (1) depend on the algebraic properties of the operator ring $R[\theta_1, \ldots, \theta_q]$. To determine the structure of this ring, in the remainder of this section we relate it to the ring of polynomials over R in q symbols.

Let $R[s_1, s_2, \ldots, s_q]$ denote the ring of polynomials in the symbols $s_1, s_2, \ldots, s_q$ with coefficients in R, and define the map

$$\rho : R[s_1, \ldots, s_q] \to R[\theta_1, \ldots, \theta_q] : \alpha(s) \mapsto \alpha(\theta)$$

The map ρ is a surjective ring homomorphism, and thus $R[\theta_1, \ldots, \theta_q]$ is isomorphic to the factor ring $R[s_1, \ldots, s_q] / \ker \rho$ where $\ker \rho = \{\alpha(s) : f(\alpha(s)) = 0\}$. Then since $R[\theta_1, \ldots, \theta_q]$ is a homomorphic image of the ring $R[s_1, \ldots, s_q]$ which is Noetherian, it follows that $R[\theta_1, \ldots, \theta_q]$ is also a Noetherian ring. Summing up these results, we have

<u>Proposition 1</u>: Given any finite list $\theta_1, \ldots, \theta_q$ of elements belonging to V, the operator ring $R[\theta_1, \ldots, \theta_q]$ is a Noetherian (integral) domain which is isomorphic to $R[s_1, \ldots, s_q] / \ker \rho$.

The elements $\theta_1,\ldots,\theta_q$ are said to be algebraically independent over R (viewed as a subring of V) if the map $\alpha(s) \mapsto \alpha(\theta)$ is an isomorphism, in which case $R[\theta_1,\ldots,\theta_q]$ is isomorphic to the polynomial ring $R[s_1,\ldots,s_q]$. Hence, $\theta_1,\ldots,\theta_q$ are algebraically independent over R if and only if there does not exist a nonzero polynomial $\alpha(s)$ such that $\alpha(\theta) = 0$. For $q = 1$, an element $\theta \in V$, algebraically independent over R, is said to be transcendental over R. Examples of transcendental elements are given in the following

Proposition 2: For any $a \in R$, $a \neq 0$, δ_a is transcendental over R, and $p = \delta_o^{(1)}$ is transcendental over R.

Proof: For $a \neq 0$, δ_a is transcendental over R since the supports of $\delta_a^o = \delta_o, \delta_a, \delta_a^2 = \delta_{2a}, \ldots, \delta_a^n = \delta_{na}$ do not intersect for any positive integer n. Now suppose $p = \delta_o^{(1)}$ is not transcendental over R. Then there exists a nonzero polynomial $\alpha(s) = \sum_{i=o}^{n} a_i s^i \in R[s]$ such that

$\alpha(p) = \sum_{i=o}^{n} a_i p^i = 0$. We can take $a_n \neq 0$. Let h = Heaviside function. Then $h^n * \alpha(p) = 0$ implies that $a_n \delta_o = \sum_{i=o}^{n-1} a_n h^{n-i}$ since $\delta_o^{(i)} * h^n = h^{n-i}$.

But this is impossible since the support of $\sum_{i=o}^{n-1} a_n h^{n-i} = \sum_{i=o}^{n-1} \frac{a_n t^{n-i-1}}{(n-i-1)!} h$ is not equal to the origin. Hence we have a contradiction which gives the desired result.

Corollary: Given $a_1, a_2, \ldots, a_r \in R$ with $a_i \neq 0$, all i, and with $a_i \neq ma_j$ for $i \neq j$ and any integer m, then $p, \delta_{a_1}, \ldots, \delta_{a_r}$ are algebraically independent over R.

Proof: Since the supports of any two elements $p^n * \delta_{a_1}^{n_1} * \ldots * \delta_{a_r}^{n_r}$ and $p^{\bar{n}} * \delta_{a_1}^{\bar{n}_1} * \ldots * \delta_{a_n}^{\bar{n}_r}$ do not intersect for any positive integers $n, n_1, \ldots, n_r, \bar{n}, \bar{n}_1, \ldots, \bar{n}_r$ with $n_i \neq \bar{n}_i$ for at least one i, it follows from Proposition 2 that $p, \delta_{a_1}, \ldots, \delta_{a_r}$ are algebraically independent over R.

In many cases of interest, the elements $\theta_1, \theta_2, \ldots, \theta_q$ generating the operator ring $R[\theta_1, \ldots, \theta_q]$ are algebraically independent over R. For example, most delay-differential systems can be specified by the set of equations (1) where $\theta = p, \delta_{a_1}, \ldots, \delta_{a_r}$ with the a_i as in the above corollary.

3. Quotient Field Operations

Given the finite set of equations (1), in this section we consider the existence and construction of solutions by utilizing operations in quotient fields.

Since the ring V of distributions is an integral domain, the smallest field in which V can be embedded is its quotient field, denoted by Q. The elements of Q are equivalence classes whose representatives are denoted by $\frac{u}{v}$ where $u, v \in V$, $v \neq 0$. With respect to this notation, any two elements $\frac{u}{v}, \frac{u'}{v'} \in Q$ are equal if and only if $u * v' = v * u'$. The operations of addition and multiplication in the field Q are defined by

$$\frac{u}{v} + \frac{u'}{v'} = \frac{u*v' + v*u'}{v*v'}$$

$$\frac{u}{v} \cdot \frac{u'}{v'} = \frac{u*u'}{v*v'}$$

The ring V is embedded in its quotient field by the map $\mathcal{J}:V \to Q:v \mapsto \frac{v}{\delta_o}$

Usually, $\mathcal{J}(v)$ will be denoted by v.

Now let $\theta_1,\theta_2,\ldots,\theta_q$ be a finite list of fixed elements in V as before. Then since the ring $R[\theta_1,\ldots,\theta_q]$ is an integral domain, the smallest field in which $R[\theta_1,\ldots,\theta_q]$ can be embedded is its quotient field, denoted by $R(\theta_1,\ldots,\theta_q)$. Clearly, $R(\theta_1,\ldots,\theta_q)$ is a subfield of Q, and in fact it is the smallest subfield of Q containing $\theta_1,\ldots,\theta_q$ and R (viewed as a subfield of Q).

For positive integers m and k, let $R[\theta_1,\ldots,\theta_q]^{k\times m}$ denote the $R[\theta_1,\ldots,\theta_q]$-module of $k\times m$ matrices over $R[\theta_1,\ldots,\theta_q]$, and let V^m denote the free V-module of m-element column vectors over V. With respect to this notation, the set of equations (1) can be written in the following matrix form:

$$A(\theta)*y = B(\theta)*u \tag{2}$$

where $A(\theta) = \left(\alpha_{ij}(\theta)\right) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$,

$B(\theta) = \left(\beta_{ij}(\theta)\right) \in R[\theta_1,\ldots,\theta_q]^{k\times m}$, $y = (y_1,\ldots,y_k)^{TR} \in V^k$,

and $u = (u_1,\ldots,u_m)^{TR} \in U^m$ (TR = transpose).

Let $R(\theta_1,\ldots,\theta_q)^{k\times k}$ denote the ring of $k\times k$ matrices over the quotient field $R(\theta_1,\ldots,\theta_q)$. By a well-known result in matrix algebra, $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$ has a (unique) inverse in the matrix ring $R(\theta_1,\ldots,\theta_q)^{k\times k}$ if and only if the determinant of $A(\theta)$, denoted by det $A(\theta)$, is not zero. If det $A(\theta) \neq 0$, we denote the inverse of $A(\theta)$ by $A(\theta)^{-1}$. Finally, letting $V^{k\times m}$ denote the V-module of $k\times m$ matrices over V, we have the following results on the existence of solutions of (2).

Proposition 3: If det $A(\theta) \neq 0$, for any $u \in V^m$ (2) has the unique solution $y = A(\theta)^{-1}\cdot B(\theta)*u \in V^k$ if and only if $A(\theta)^{-1}\cdot B(\theta) \in V^{k\times m}$.

Proof: If : Viewing (2) as a set of equations over Q, we obtain the solution $y = A(\theta)^{-1}\cdot B(\theta)\cdot u \in Q^k$. Then since $A(\theta)^{-1}\cdot B(\theta) \in V^{k\times m}$, $y = A(\theta)^{-1}\cdot B(\theta)*u \in V^k$, all $u \in V^m$. Uniqueness of y follows from the uniqueness of matrix operations over the field Q.

Only if: Let $W(\theta) = (w_{ij}(\theta)) = A(\theta)^{-1}\cdot B(\theta)$, and for every $j = 1,2,\ldots,m$, let e_j be the element of V^m all of whose components are zero except for the j^{th} which is equal to δ_o.

Then since $y = W(\theta)\cdot e_j = (w_{1j},\ldots,w_{kj})^{TR} \in V^k$ for all j, we have that $W(\theta) \in V^{k\times m}$.

Corollary 1: If det $A(\theta)$ has an inverse $(\det A(\theta))^{-1} \in V$, then for any $u \in V^m$, (2) has a unique solution y in V^k, given by

$$y = (\det A(\theta))^{-1}*\tilde{A}(\theta)*B(\theta)*u \tag{3}$$

where $\tilde{A}(\theta)$ is the transpose of the matrix of cofactors of $A(\theta)$.

Proof: Let $V^{k\times k}$ denote the ring of $k\times k$ matrices over the ring V. Then $R[\theta_1,\ldots,\theta_q]^{k\times k}$ is a subring of $V^{k\times k}$ and by standard results in matrix algebra, $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$ has a (unique) inverse in $V^{k\times k}$ if and only if det $A(\theta)$ is a unit (invertible element) in V. If det $A(\theta)$ is a unit in V, then $A(\theta)^{-1} = (\det A(\theta))^{-1}*\tilde{A}(\theta)$ and $A(\theta)^{-1}*B(\theta) \in V^{k\times m}$. Hence, by the above proposition, the corollary is proved.

Corollary 2: If $B(\theta)$ has an $m\times k$ right inverse over V, then for any $u \in V^m$ (2) has a unique solution $y \in V^k$ if and only if $\det A(\theta)$ is a unit in V.

Proof: The if part follows from proposition 3. Now suppose that (2) has a unique solution $y \in V^k$ for any $u \in V^m$, and for every $i = 1,2,\ldots,k$, let w_{ij} denote the i^{th} component of the solution when $u = e_j$, $j = 1,2,\ldots,m$. Then $A(\theta)*W = B(\theta)$ where $W = (w_{ij})$, and if $B(\theta)$ has a right inverse M over V, we have that $A(\theta)*(W*M) = I_k$ where I_k is the $k\times k$ identity matrix. Thus, $A(\theta)$ is a unit in $V^{k\times k}$ which implies that $\det A(\theta)$ is a unit in V.

Note that if $A(\theta)^{-1}\cdot B(\theta) \in V^{k\times m}$, the system given by (2) can be represented by the restriction to U^m of the input/output operator $f:V^m \to V^k : v \mapsto A(\theta)^{-1}\cdot B(\theta)*v$. With respect to the free V-module structure on V^m and V^k, f is a V-module homomorphism and $A(\theta)^{-1}\cdot B(\theta)$ can be viewed as the matrix representation of f relative to the standard bases in V^m and V^k.

As seen from (3), solutions of (2) can be computed by first finding the inverse in V of $\det A(\theta)$ if it exists. To simplify this computation we can utilize the property that the operator ring $R[\theta_1,\ldots,\theta_q]$ is Noetherian. As a consequence, every nonunit of the ring $R[\theta_1,\ldots,\theta_q]$ can be written as a finite product of factors that are irreducible in $R[\theta_1,\ldots,\theta_q]$. Further, if $\theta_1,\ldots,\theta_q$ are algebraically independent over R, then $R[\theta_1,\ldots,\theta_q]$ is a unique factorization domain, and hence factorizations into irreducible elements are unique (the proof of these statements can be found in Zariski and Samuel [2]). Now as a consequence of the commutative ring structure on V, we have the following

Lemma: Let $\det A(\theta)$ be a nonunit in $R[\theta_1,\dots,\theta_q]$ and let $\det A(\theta) = \pi_1 * \pi_2 * \dots * \pi_\ell$ be a decomposition of $\det A(\theta)$ into irreducible factors. Then $\det A(\theta)$ is a unit in V if and only if each π_i has an inverse π_i^{-1} in V, in which case $(\det A(\theta))^{-1} = \pi_1^{-1} * \pi_2^{-1} * \dots * \pi_\ell^{-1}$.

Combining Corollary 1 of Proposition 3 and the above Lemma, we have

Theorem 1: If $\det A(\theta)$ is a non-unit in $R[\theta_1,\dots,\theta_q]$ and if $\det A(\theta) = \pi_1 * \pi_2 * \dots * \pi_\ell$ is a decomposition of $\det A(\theta)$ into irreducible factors with each π_i having inverse π_i^{-1} in V, then for any $u \in V^m$, (2) has the unique solution

$$y = (\pi_1^{-1} * \pi_2^{-1} * \dots * \pi_\ell^{-1}) * \tilde{A}(\theta) * B(\theta) * u. \tag{4}$$

A fundamental point here is that in determining the existence of solutions of (2) assuming $\det A(\theta) \neq 0$, it is necessary to consider only the invertibility in V of the irreducible elements of the operator ring $R[\theta_1,\dots,\theta_q]$. In particular, combining the above results, we obtain:

Theorem 2: If every irreducible element of $R[\theta_1,\dots,\theta_q]$ is a unit in V, then the quotient field $R(\theta_1,\dots,\theta_q)$ is contained in V, and for any $A(\theta) \in R[\theta_1,\dots,\theta_q]^{k\times k}$ and $B(\theta) \in R[\theta_1,\dots,\theta_q]^{k\times m}$ with $\det A(\theta) \neq 0$, (2) has the unique solution (3) for all $u \in V^m$.

Note that if $R(\theta_1,\dots,\theta_q) \subset V$, then V is a linear space over $R(\theta_1,\dots,\theta_q)$ with the multiplication $R(\theta_1,\dots,\theta_q)\times V \to V: \left(\frac{\alpha}{\beta}, v\right) \mapsto \beta^{-1} * \alpha * v$. In this case, to solve (2) we first can simplify the set of equations by reducing $A(\theta)$ to upper or lower triangular form via row operations. For the case $q = 1$ and $\theta =$ derivative operator, this approach is similar to that given by Blomberg et al [3].

Unfortunately, even if det $A(\theta)$ can be decomposed into irreducible factors, the actual computation of solutions via (4) is usually quite difficult as a result of the complexity of convolution operations. To simplify the problem of computation, in the next section we consider an algebraic procedure that extends the classical operational calculus.

4. Computation of Solutions

Given a system specified by the input/output equations

$$A(\theta)*y = B(\theta)*u \tag{5}$$

where $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$ with det $A(\theta) \neq 0$, and $B(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times m}$, the expression of $A(\theta)^{-1}\cdot B(\theta)$ as an element of $R(\theta_1,\ldots,\theta_q)^{k\times m}$ is referred to as the operational transfer function matrix of the system. In correspondence with the standard terminology, the expression of $A(\theta)^{-1}\cdot B(\theta)$ as an element of $V^{k\times m}$ (assuming that $A(\theta)^{-1}\cdot B(\theta)$ is contained in $V^{k\times m}$) is called the impulse response frunction matrix of the system.

Since $R(\theta_1,\ldots,\theta_q)$ is a field, the inverse of $A(\theta)$ over $R(\theta_1,\ldots,\theta_q)$ can be computed by the usual techniques of inverting matrices over a field. Hence the operational transfer function matrix $A(\theta)^{-1}\cdot B(\theta)$ can be determined from (5) by using standard procedures. Furthermore, if the input u can be expressed as an element of $R(\theta_1,\ldots,\theta_q)^m$, then as an element of $R(\theta_1,\ldots,\theta_q)^k$, the output y is readily computed. The main difficulty in obtaining solutions of (5) via this procedure is expressing y as an element of V^k. Hence the central problem is expressing elements of $R(\theta_1,\ldots,\theta_q)$ as elements in V (when possible), which we shall refer to as the process of inversion. We now consider an algebraic method which simplifies this problem.

First, let $q = 1$ and let $\theta \in V$ be any fixed transcendental element over R. Then as discussed in Section 2, the operator ring $R[\theta]$ is isomorphic to the polynomial ring over R in one symbol. Hence $R[\theta]$ is a principal ideal domain, and from classical results in algebra (see MacLane and Birkhoff [4, page 186]) the only irreducible elements of $R[\theta]$ are linear elements $a\theta + b$ and quadratic elements $a\theta^2 + b\theta + c$ with negative discriminant $b^2 - 4ac < 0$. Therefore if these elements are units in V, then the quotient field $R(\theta)$ is contained in V, and for any $A(\theta) \in R[\theta]^{k\times k}$ with $\det A(\theta) \neq 0$ and $B(\theta) \in R[\theta]^{k\times m}$, the set of equations $A(\theta)*y = B(\theta)*u$ has a unique solution $y \in V^k$ for any $u \in V^m$.

The inverse in V (if it exists) of linear elements $a\theta + b \in R[\theta]$ can be determined by the usual techniques such as taking the limit of the sequence of partial sums obtained from a power series expansion of $\frac{1}{a\theta + b}$. Moreover, since the field of complex numbers C is the splitting field of $R[\theta]$, the problem of inverting quadratic elements with negative discriminant reduces to the problem of inverting linear elements over C. (We must then take V to be the space of C-valued distributions with supports bounded on the left.)

Since $R[\theta]$ is a principal ideal domain, to compute the inverse in V of elements in $R(\theta)$, we can apply the method of partial fraction expansions which reduces the problem of inversion to computing the inverses of linear and quadratic elements (or linear elements over C). If $\theta = \delta_o^{(1)} =$ derivative operator, this procedure yields the classical operational calculus. The main point here is that operational techniques apply to any class of systems described by operators in $R[\theta]$ with $\theta \in V$ transcendental over R.

Now let us consider the problem of inverting elements in $R(\theta_1,\ldots,\theta_q)$ when $q > 1$ and $\theta_1,\ldots,\theta_q$ are algebraically independent over R. For any fixed $i = 1,2,\ldots,q$, let $\theta-\theta_i$ denote the list $\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_q$ so that $R[\theta-\theta_i][\theta_i]$ denotes the ring of polynomial operators in θ_i over the ring $R[\theta-\theta_i]$. The ring $[\theta_1,\ldots,\theta_q]$ is isomorphic to $R[\theta-\theta_i][\theta_i]$ and the quotient field $R(\theta_1,\ldots,\theta_q)$ is isomorphic to $R(\theta-\theta_i)(\theta_i)$. Hence the elements of $R(\theta_1,\ldots,\theta_q)$ can be viewed as elements in the quotient field of the ring $R(\theta-\theta_i)[\theta_i]$. Then since $R(\theta-\theta_i)[\theta_i]$ is a principal ideal domain, any element of $R(\theta_1,\ldots,\theta_q)$, viewed as an element of $R(\theta-\theta_i)(\theta_i)$ for some fixed i, can be decomposed via a partial fraction expansion as follows.

Let Ω denote the set of monic irreducible polynomials in $R(\theta-\theta_i)[\theta_i]$. After multiplication by elements in $R(\theta-\theta_i)$ if necessary, the elements of Ω actually belong to $R[\theta-\theta_i][\theta_i]$ since if α is an irreducible polynomial in $R[\theta-\theta_i][\theta_i]$, it is also an irreducible polynomial in $R(\theta-\theta_i)[\theta_i]$ (see Zariski and Samuel [2, page 102]). Then given any element $\frac{\alpha}{\beta} \in R(\theta_1,\ldots,\theta_q)$, $\frac{\alpha}{\beta}$ has a unique decomposition

$$\frac{\alpha}{\beta} = \sum_{\omega \in \Omega} \frac{\pi_\omega}{\omega^{j(\omega)}} + \gamma \tag{6}$$

where π_ω, $\gamma \in R(\theta-\theta_i)[\theta_i]$, $j(\omega)$ are non-negative integers, $\pi_\omega=0$ if $j(\omega)=0$, π_ω is relatively prime to ω if $j(\omega) > 0$, and $\deg \pi_\omega < \deg \omega^{j(p)}$ if $j(p) > 0$. (See Lang [5, page 125].) The expression (6) can be decomposed further by viewing the π_ω as polynomials in $R(\theta-\theta_j)[\theta_j]$, $i \neq j$, and then applying the partial fraction expansion to each π_ω, and so on.

This procedure can greatly simplify the problem of inverting elements of $R(\theta_1,\ldots,\theta_q)$. For illustrative purposes, we present the following.

Example: Consider the delay-differential system given by the following input/output equations

$$\frac{d^2y_1(t)}{dt^2} + \frac{dy_1(t-1)}{dt} + y_2(t-1) + y_2(t) = \frac{du_1(t-1)}{dt} + 2u_1(t) + \frac{2du_2(t-2)}{dt}$$

$$\frac{-dy_1(t)}{dt} - y_1(t-1) + y_2(t) = -u_1(t-1) - u_2(t-2).$$

If we let $d = \delta_1$ and $p = \delta_o^{(1)}$, then the system can be specified in terms of operators belonging to the ring $R[d,p]$ with d and p algebraically independent over R (by the corollary to proposition 2).

In matrix form, we have $A(p,d)*y = B(p,d)*u$ where

$$A(p,d) = \begin{pmatrix} p^2+dp & d+1 \\ -p-d & 1 \end{pmatrix}, \quad B(p,d) = \begin{pmatrix} dp+2 & 2d^2p \\ -d & -d^2 \end{pmatrix}$$

Computing the inverse $A(p,d)^{-1}$ of $A(p,d)$ over $R(p,d)$ and multiplying by $B(p,d)$, we obtain

$$A(p,d)^{-1}\cdot B(p,d) = \frac{1}{p^2+(2d+1)p+d^2+d} \begin{pmatrix} dp+d^2+d+2 & 2d^2p+d^3+d^2 \\ 2p+2d & d^2p^2+d^3p \end{pmatrix}$$

which is the operational transfer function of the system (we are omitting the $*$ for convolution). Now let $u = \left(-e^{-t}h(t),\ h(t)\right)^{TR}$ where $h(t)$ = Heaviside function. As an element of $R(p,d)^2$, we have that $u = \left(\frac{-1}{p+1}, \frac{1}{p}\right)^{TR}$.

Viewing $\det A(p,d) = p^2+(2d+1)p+d^2+d$ as a polynomial in $R(d)[p]$ and applying the quadratic formula, we obtain $\det A(p,d) = (p+d)(p+d+1)$.

Then

$$y = A(p,d)^{-1} \cdot B(p,d) \cdot u = \begin{pmatrix} \dfrac{(2d^2-d)p^2+(d^3+2d^2-d-2)p+d^3+d^2}{p(p+1)(p+d)(p+d+1)} \\ \\ \dfrac{d^2p+d^2-2}{(p+1)(p+d)(p+d+1)} \end{pmatrix}$$

To invert y, we shall view the components of y as elements in the quotient field of R(d)[p] and expand by partial fractions. This gives

(7) $$y_1 = \frac{d}{p} - \frac{\frac{d^2+2}{d(d-1)}}{p+1} + \frac{\frac{d^3-2d^2+2d+2}{d-1}}{p+d} - \frac{\frac{d^3+2}{d}}{p+d+1}$$

(8) $$y_2 = - \frac{\frac{2}{d(d-1)}}{p+1} - \frac{\frac{-d^3+d^2-2}{d-1}}{p+d} + \frac{\frac{-d^3-2}{d}}{p+d+1}$$

Now y_1 and y_2 can be decomposed further by performing the following expansions

(9) $$\frac{d^2+2}{d(d-1)} = 1 - \frac{2}{d} + \frac{3}{d-1}$$

(10) $$\frac{2}{d(d-1)} = \frac{-2}{d} + \frac{2}{d-1}$$

From (7) - (10), it is seen that the inversion of y_1 and y_2 reduces to the problem of inverting $\frac{1}{p+1}$, $\frac{1}{p+d}$, $\frac{1}{d-1}$, and $\frac{1}{p+d+1}$. Via power series expansion, we obtain

$$\frac{1}{p+1} \equiv e^{-t}h(t)$$

$$\frac{1}{d-1} \equiv - \sum_{n=o}^{\infty} \delta_n$$

$$\frac{1}{p+d} \equiv \sum_{n=o}^{\infty} \frac{(n-t)^n}{n!} h(t-n) \triangleq f(t)$$

$$\frac{1}{p+d+1} \equiv \sum_{n=o}^{\infty} \frac{(n-t)^n}{n!} e^{-(t-n)} h(t-n) \triangleq g(t)$$

It can be easily checked that $\sum_{n=o}^{\infty} \delta_n \in V$, and that the other functions are locally integrable and thus also belong to V. Hence, a solution $y = (y_1, y_2)^{TR} \in V^2$ exists, and is given by

$$y_1 = h(t-1) - e^{-t}h(t)+2e^{-(t+1)}h(t+1)+3\sum_{n=o}^{\infty} e^{-(t-n)}h(t-n)$$

$$- \sum_{n=o}^{\infty}\left[f(t-n-3)-2f(t-n-2)+2f(t-n-1)+2f(t-n)\right] -g(t-2)-g(t+1)$$

$$y_2 = 2e^{-(t+1)}h(t+1)-2\sum_{n=o}^{\infty} e^{-(t-n)}h(t-n)+\sum_{n=o}^{\infty}\left[f(t-n-3)-f(t-n-2)+2f(t-n)\right]$$

$$-g(t-2)-2g(t+1)$$

It is clear from this example that the success of the above algebraic procedure in simplifying the computation of solutions depends on the decomposibility of det $A(\theta)$. When det $A(\theta)$ is decomposable into factors of low degree, this technique of computing solutions compares quite favorably with classical procedures for solving operational-differential equations (see Bellman and Cooke [6]). We also mention that when the generators of the operator ring are specified, the algebraic framework could be applied to equations with initial conditions.

5. Internal Description

Consider a system specified by the input/output operator $f:U^m \to V^m: u \mapsto W * u$ where $W \in V^{k\times m} \cap R(p,\theta_1,\ldots,\theta_r)^{k\times m}$. If the inputs and resulting outputs are regular distributions (generated by locally integrable functions), the classical state space representation of the system (if one exists) is given by

$$(11) \qquad \frac{dx(t)}{dt} = Fx(t) + Gu(t)$$
$$y(t) = Hx(t)$$

where F,G,H are linear maps, $u(t) \in R^m$, $y(t) \in R^k$, and the state x(t) at time t belongs to some locally convex R-linear topological space X, called the state space. If the system contains infinite elements, then X is an infinite-dimensional linear space, and thus the matrix representations of the linear maps F,G,H have infinite size.

To circumvent this infinite dimensionality, we consider the class of "hereditary systems" in which $X = R^n$, $n < \infty$, and the derivative of x(t) at time t depends on x(t) and u(t) over a past interval $(t-\tau, t]$ for some fixed $\tau > 0$. (Here x(t) is no longer the state in the classical sense.) An example of a finite hereditary system ($\tau < \infty$) is a delay-differential system of the form

$$(12) \qquad \frac{dx(t)}{dt} = \sum_{i=1}^{\ell} F_i x(t-b_i) + \sum_{i=1}^{p} G_i u(t-c_i)$$
$$y(t) = \sum_{i=1}^{q} H_i x(t-d_i)$$

where $b_i, c_i, d_i \geq 0$ and F_i, G_i, H_i are matrices over R of size $n \times n$, $n \times m$, and $k \times n$, respectively.

Representations similar to (12) have been used extensively to study the internal properties, such as control, of delay-differential systems. (For example, see Oğuztöreli [7].) Usually, in the literature $y(t) = x(t)$, but it is reasonable to consider situations in which there are also time delays between x(t) and the output y(t).

Our objective here is to extend representations of the form (12) to a general class of operational-differential systems and to do this in terms of operators belonging to the ring $R[p,\theta_1,\ldots,\theta_r]$. First note that we can write (12) in the form

$$\begin{aligned} \frac{dx(t)}{dt} &= (F*x)(t) + (G*u)(t) \\ y(t) &= (H*x)(t) \end{aligned} \tag{13}$$

where F,G,H are matrices of size $n\times n$, $n\times m$, $k\times n$ over the operator ring $R[\delta_{a_1},\ldots,\delta_{a_r}]$ for some $a_i \in R$. From (13) we obtain the desired generalization as follows.

Let $\mathcal{D}$ denote the space of "testing functions" associated with the space of distributions V. Given any positive integer n and $v = (v_1,\ldots,v_n)^{TR} \in V^n$, we define $v(\varphi)$, $\varphi \in \mathcal{D}$, by $v(\varphi) = (v_1(\varphi),\ldots,v_n(\varphi)) \in R^n$. Given $x \in V^n$, we let $p*x \equiv \frac{dx}{dt}$ denote the operation of $p = \delta_o^{(1)}$ on x in the V-module structure on V^n. We then have the following:

Definition: An m-input terminal k-output terminal operational-differential system over $R[p,\theta_1,\ldots,\theta_r]$ is a triple (F,G,H) of $n\times n$, $n\times m$, $k\times n$ matrices over $R[\theta_1,\ldots,\theta_r]$ such that $(pI-F)^{-1}\cdot G \in V^{n\times m}$, together with the following set of operational-differential equations in the sense of distributions

$$\begin{aligned} (p*x)(\varphi) \equiv \frac{dx(\varphi)}{dt} &= (F*x)(\varphi) + (G*u)(\varphi) \\ y(\varphi) &= (H*x)(\varphi) \end{aligned} \tag{14}$$

where $u \in U^m$, $y \in V^k$, and $x \in S^n$, S = linear subspace of V. The integer n is called the size of the system.

The set of equations (14) represents a hereditary system in a generalized sense if the components of F,G,H have their supports contained in $[o,\infty)$. Furthermore, (14) is a finite hereditary system if the elements of F,G,H have compact support contained in $[o,\infty)$, which will be the case if the supports of $\theta_1,\ldots,\theta_r$ are compact and contained in $[o,\infty)$. This latter condition implies that the infinite devices comprising the system have impulse responses with compact support $\subset [o,\infty)$. In addition to ideal delay lines, many types of dispersive lines satisfy this condition.

Note that by placing suitable constraints on U and F,G,H, we could restrict our attention to operational-differential equations defined in the ordinary sense (that is, we can replace φ by t). We could then consider extending the theory of hereditary systems by using the framework given by (14). However, the fundamental problem of interest here is constructing representations of the form (14) from operational transfer functions $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m}$.

First, solving (14) over the quotient field Q of V, we have that $x = (pI-F)^{-1}\cdot G*u \in V^n$ since $(pI-F)^{-1}\cdot G \in V^{n\times m}$. Then $y=H\cdot(pI-F)^{-1}\cdot G*u \in V^k$ since H is over $R[\theta_1,\ldots,\theta_r] \subset V$. Hence,
$H\cdot(pI-F)^{-1}\cdot G \in R(p,\theta_1,\ldots,\theta_r)^{k\times m} \cap V^{k\times m}$ is the operational transfer function matrix of the system.

We then have the following:

<u>Definition of Realization</u>: Given an operator
$f:U^m \to V^k: u \mapsto W*u$, $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m} \cap V^{k\times m}$, a realization of f over $R[\theta_1,\ldots,\theta_r]$ is a system of the form (14) with $W = H\cdot(pI-F)^{-1}\cdot G$.

In the next section we pursue the problem of constructing realizations by considering the decomposability of the operational transfer function matrix.

6. Decomposition of Transfer Functions

Again, let $\theta = \theta_1,\ldots,\theta_q$ be a finite list of elements belonging to V, and for any fixed $i = 1,2,\ldots,q$, let $\theta-\theta_i$ denote the list $\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_q$ so that $R[\theta-\theta_i]$ denotes the operator ring over R in the elements $\theta-\theta_i$. Let $R[\theta-\theta_i][\theta_i] = \{\sum_{j=o}^{n} \pi_j * \theta_i^j : \pi_j \in R[\theta-\theta_i]\}$. With the usual operations, the set $R[\theta-\theta_i][\theta_i]$ is a ring which is isomorphic to $R[\theta_1,\ldots,\theta_q]$. Finally, letting $R[\theta-\theta_i][s]$ denote the ring of polynomials over $R[\theta-\theta_i]$ in the symbol s, throughout this section we assume that the map $\xi : R[\theta-\theta_i][s] \to R[\theta-\theta_i][\theta_i] : \alpha(s) \mapsto \alpha(\theta_i)$ is a ring isomorphism so that $R(\theta-\theta_i)[\theta_i]$ is a principal ideal domain.

Given $W \in R(\theta_1,\ldots,\theta_q)^{k\times m}$, we say that W is decomposable over $R(\theta-\theta_i)$ (respectively, over $R[\theta-\theta_i]$) if there exist matrices F,G,H over $R(\theta-\theta_i)$ (respectively $R[\theta-\theta_i]$), where F is $n\times n$, G is $n\times m$, and H is $k\times n$, such that

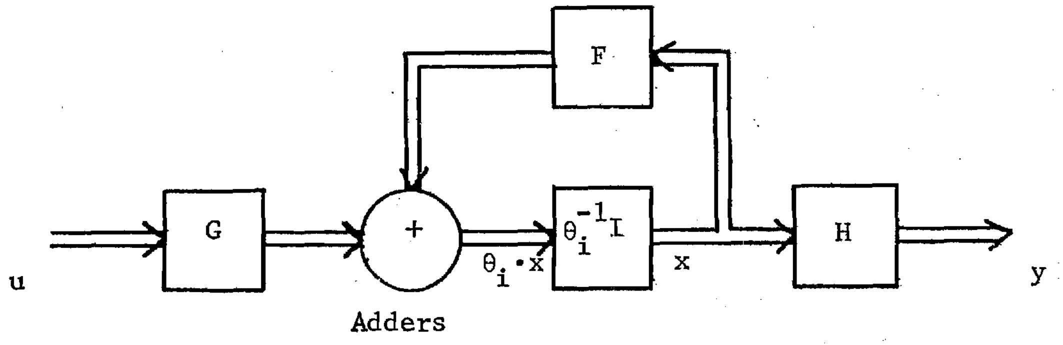
$$W = H\cdot(\theta_i I-F)^{-1}\cdot G\,. \tag{15}$$

The integer n is called the size of the decomposition. A decomposition $(F,G,H)_n$ is said to be reduced if n is minimal among all possible decompositions of the form (15).

A decomposition $(F,G,H)_n$ over $R(\theta-\theta_i)$ or $R[\theta-\theta_i]$ yields the following set of operational equations

$$\begin{aligned} \theta_i\cdot x &= F\cdot x + G\cdot u \\ y &= H\cdot x \end{aligned} \tag{16}$$

where $u \in U^m$, and in general $x \in Q^n$, $y \in Q^k$ where Q is the quotient field of V. If θ_i is a unit in V, (16) corresponds to a system with the following "wiring diagram".



This diagram illustrates that in a decomposition over $R(\theta-\theta_i)$ or $R[\theta-\theta_i]$ all devices with impulse response θ_i^{-1} are "extracted". However, in a decomposition over $R(\theta-\theta_i)$ the elements of F,G,H may not be distributions (i.e., they belong to Q) or if they are, their supports may not be contained in $[o,\infty)$ even if the supports of the elements in the list $\theta-\theta_i$ are contained in $[o,\infty)$.

The main interest here in decompositions over $R(\theta-\theta_i)$ is that this problem can be viewed as a first step in determining decompositions over $R[\theta-\theta_i]$ which, in turn, for the case $\theta_i = p = \delta_o^{(1)}$ leads to the construction of realizations as defined in the preceding section. In particular, if $(F,G,H)_n$ is a decomposition of $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m}$ over $R[\theta-p]$, then $(F,G,H)_n$ defines a realization of the operator $f:U^m \to V^k: u \mapsto W*u$ as given by (14) if $(pI-F)^{-1}\cdot G \in V^{n\times m}$. This latter condition is satisfied if det $(pI-F)$ is a unit in V which is always the case if $R(p,\theta_1,\ldots,\theta_r)$ is contained in V.

Although we are primarily interested in the case $\theta_i = \delta_o^{(1)}$, we consider the construction of decompositions over $R(\theta-\theta_i)$ and $R[\theta-\theta_i]$ for any θ_i such that $R(\theta-\theta_i)[\theta_i]$ is a principal ideal domain (abbreviated pid). The main result on decompositions over $R(\theta-\theta_i)$ is given in the following:

<u>Theorem 3</u>: $W = \left(\frac{\alpha_{ij}}{\beta_{ij}}\right) \in R(\theta_1,\ldots,\theta_q)^{k\times m}$ is decomposable over $R(\theta-\theta_i)$ if the degree of α_{ij} is less than the degree of β_{ij} for any fixed i,j when α_{ij} and β_{ij} are viewed as elements in $R[\theta-\theta_i][\theta_i]$.

Several constructive proofs of this theorem can be given. The first one that we consider is based on the invariant factor theorem for pids. Let W satisfy the hypothesis of the theorem. Since $R[\theta-\theta_i][\theta_i]$ is contained in $R(\theta-\theta_i)[\theta_i]$, the elements of W can be viewed as elements in the quotient field of the ring $R(\theta-\theta_i)[\theta_i]$ which is a principal ideal domain. If ψ is the least common denominator of W as matrix over $R(\theta-\theta_i)(\theta_i)$, then ψW is a matrix over $R(\theta-\theta_i)[\theta_i]$. Since $R(\theta-\theta_i)[\theta_i]$ is a pid, by the invariant factor theorem, we can reduce ψW to diagonal form from which the matrices F,G,H can be computed by using Kalman's procedure [8]. For the details of this procedure along with an example, see Kamen [9].

Other proofs of Theorem 3 are based on a Hankel matrix sequence which is generated in the following manner. Again viewing W as a matrix over the quotient field of the pid $R(\theta-\theta_i)[\theta_i]$, by long division we can expand each element of W into a formal power series in θ_i^{-1} with coefficients in the field $R(\theta-\theta_i)$. This yields

$$W = \sum_{\ell=1}^{\infty} A_\ell \theta_i^{-\ell}, \quad A_\ell \in R(\theta-\theta_i)^{k\times m}, \quad \ell = 1,2,3,\ldots$$

We then define a sequence of Hankel matrices for W by

$$\Gamma_{i,j} = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_{j-1} \\ A_2 & A_3 & A_4 & \cdots & A_j \\ A_3 & A_4 & A_5 & \cdots & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ A_{i-1} & \cdot & \cdot & \cdots & A_{i+j-1} \end{pmatrix}$$

Now if a decomposition of W over $R(\theta-\theta_i)$ exists, then $W = H\cdot(\theta_i I-F)^{-1}\cdot G$. Expanding $(\theta_i I-F)^{-1}$, we obtain

$$H\cdot(\theta_i I-F)^{-1}\cdot G = \sum_{\ell=1}^{\infty} H*F^{\ell-1}*G\cdot\theta_i^{-\ell}.$$

Hence W is decomposable over $R(\theta-\theta_i)$ if there exists matrices F,G,H such that $A_\ell = H*F^{\ell-1}*G$, $\ell = 1,2,3,\ldots$ Again let ψ = least common denominator of W. Then it follows from the results of Ho [10] that W has a reduced decomposition over $R(\theta-\theta_i)$ of size equal to the rank of $\Gamma_{\sigma,\sigma}$ where σ is the degree of ψ as an element of $R[\theta-\theta_i][\theta_i]$. The matrices F,G,H can be computed from $\Gamma_{\sigma,\sigma+1}$ by using Ho's algorithm. The details have been carried by Newcomb [11] for the case in which W is a rational function in several complex variables.

Another procedure for computing F,G,H is to use Silverman's formulas as derived by Rouchaleau [12]: Let J be a submatrix of $\Gamma_{\sigma,\sigma}$ having maximal rank n. Let K be the n×m submatrix of the first block column (the first m elementary columns) of $\Gamma_{\sigma,\sigma}$ corresponding to the rows of J. Finally, let L be the k×n submatrix of the first block row (the first k elementary rows) of $\Gamma_{\sigma,\sigma}$ corresponding to the columns of J. Then $F = J^{-1}M$, $G = J^{-1}K$, $H = L$ is a reduced decomposition where M is the n×n submatrix of $\Gamma_{\sigma,\sigma+1}$ sitting m elementary columns to the right of J. An example of this procedure is given in the following section.

Any two reduced decompositions over $R(\theta-\theta_i)$ are unique in the following sense.

Theorem 4: If $(F,G,H)_n$ and $(\hat{F},\hat{G},\hat{H})_n$ are two reduced decompositions of W over $R(\theta-\theta_i)$, then there exists an $n\times n$ invertible matrix T over $R(\theta-\theta_i)$ such that

$$\hat{F} = TFT^{-1}, \quad \hat{G} = TG, \quad \hat{H} = HT^{-1}.$$

This theorem can be proved by applying the construction of Kalman [8] to the pid $R(\theta-\theta_i)[\theta_i]$. We omit the details.

We now consider decompositions over $R[\theta-\theta_i][\theta_i]$. Here we utilize the algebraic theory of linear discrete-time systems over commutative rings as developed by Rouchaleau [12] and Rouchaleau, Wyman, and Kalman [13]. The main contribution of this work is the application of this algebraic theory to operational systems in continuous-time.

Theorem 5: If $W = \left(\frac{\alpha_{ij}}{\beta_{ij}}\right) \in R(\theta_1,\ldots,\theta_q)^{k\times m}$ is decomposable over $R(\theta-\theta_i)$ and if W has a common denominator which is monic when viewed as an element of $R[\theta-\theta_i][\theta_i]$, then W is decomposable over $R[\theta-\theta_i]$.

To prove Theorem 5, we again view W as a matrix over $R(\theta-\theta_i)(\theta_i)$ and then expand W into a power series

$$W = \sum_{\ell=1}^{\infty} A_\ell \theta_i^{-\ell}, \quad A_\ell \in R(\theta-\theta_i)^{k\times m}, \quad \ell = 1,2,3,\ldots$$

But since W has a common denominator which is a monic polynomial in $R[\theta-\theta_i][\theta_i]$, for all $\ell = 1,2,3,\ldots$, A_ℓ is an element of $R[\theta-\theta_i]^{k\times m}$.

Now since $R[\theta-\theta_i]$ is a Noetherian domain (by Proposition 1), it follows directly from the results of Rouchaleau, Wyman, Kalman [13] that there exists matrices F,G,H over $R[\theta-\theta_i]$ such that $A_\ell = H*F^{\ell-1}*G$, $\ell = 1,2,3,\ldots$ Hence, $W = H\cdot(\theta_i I-F)^{-1}\cdot G$ which is the desired result.

The proof given in [13] of the existence of the matrices F,G,H is fairly constructive. However, in general the decomposition obtained in this manner is not reduced, and as of yet, there are no practical procedures for computing reduced decompositions over an arbitrary Noetherian ring $R[\theta-\theta_i]$. However, when $q = 2$ and θ_1 and θ_2 are algebraically independent over R, $R[\theta-\theta_i]$ is a pid and we can apply Rouchaleau's algorithm [12] to compute reduced decompositions over $R[\theta-\theta_i]$. The procedure is as follows.

Let $(\bar{F},\bar{G},\bar{H})_n$ be a reduced decomposition of $W \in R(\theta_1,\theta_2)^{k\times m}$ over $R(\theta-\theta_i)$, computed from the matrix J in the Silverman procedure. Since $R[\theta-\theta_i]$ is a pid, it follows that there exists a $n\times n$ invertible matrix T over $R(\theta-\theta_i)$ such that $F = T^{-1}\bar{F}T$, $G = T^{-1}\bar{G}$, $H = \bar{H}T$ is a reduced decomposition of W over $R[\theta-\theta_i]$. To compute **T**, let N be the $n\times\sigma m$ submatrix of the Hankel matrix $\Gamma_{\sigma,\sigma}$ containing the same rows as J. We then find a basis for the columns of N over the pid $R[\theta-\theta_i]$:

Let π_1 be the greatest common divisor of the elements in the first row of N. There is a linear combination γ_1 over $R[\theta-\theta_i]$ of the columns of N having π_1 as first element, and for each column γ of N, there exists an $\alpha \in R[\theta-\theta_i]$ such that the first element of $\gamma-\alpha\gamma_1$ is zero. Doing this for each column of N, we obtain a matrix N_1 such that γ_1 and the columns of N_1 generate the columns of N over $R[\theta-\theta_i]$.

Applying this procedure to N_1, and so on, we obtain a matrix $(\gamma_1,\ldots,\gamma_n)$ such that $\gamma_1,\ldots,\gamma_n$ generate the columns of N over $R[\theta-\theta_i]$ and $T = J^{-1}(\gamma_1,\ldots,\gamma_n)$. An example of this construction is given in the next section.

In the general case, the question of the uniqueness of reduced decompositions over $R[\theta-\theta_i]$ appears to be difficult to answer; we leave this as an open problem. However, we do have the following result.

Theorem 6: If $q = 2$ and θ_1 and θ_2 are algebraically independent over R, then given any two reduced decompositions $(F,G,H)_n$ and $(\hat{F},\hat{G},\hat{H})_n$ of $W \in R(\theta_1,\theta_2)^{k\times m}$ over $R[\theta-\theta_i]$, there exists an $n\times n$ invertible matrix A over $R[\theta-\theta_i]$ such that $\hat{F} = AFA^{-1}$, $\hat{G} = AG$, $\hat{H} = HA^{-1}$.

This theorem can be proved by extending the constructions of Kalman [8] to polynomial rings over pids.

7. An Example

Consider a delay-differential system whose input/output operator f is given by $f:U^2 \to V^2 : u \mapsto W{*}u$ where

$$W = \frac{1}{p^2+pd}\begin{pmatrix} 2d^2p & -6 \\ -2d^3p & -2p+4d \end{pmatrix} \in R(p,d)^2,\ \ p=\delta_o^{(1)},\ \ d=\delta_1$$

It is seen that W satisfies the hypothesis of Theorem 5, and thus, it has a reduced decomposition over $R[d]$ which we now compute.
Since the degree of p^2+pd, viewed as an element of $R[d][p]$, is two, we need to consider the Hankel matrix $\Gamma_{2,2}$. Expanding the elements of W, we obtain:

$$\Gamma_{2,2} = \begin{pmatrix} 2d^2 & 0 & -2d^3 & -3 \\ -2d^3 & -2 & 2d^4 & 6d \\ -2d^3 & -3 & 2d^4 & 3d \\ 2d^4 & 6d & -2d^5 & -6d^2 \end{pmatrix}$$

The rank of $\Gamma_{2,2}$ (as a matrix over R(d)) is two. We then pick

$$J = \begin{pmatrix} 2d^2 & 0 \\ -2d^3 & -2 \end{pmatrix} \quad \text{with} \quad J^{-1} = \begin{pmatrix} \frac{1}{2d^2} & 0 \\ -d/2 & -1/2 \end{pmatrix}$$

We then have that $K = J$, $L = J$, and

$$M = \begin{pmatrix} -2d^3 & -3 \\ 2d^4 & 6d \end{pmatrix}$$

which gives the following decomposition over R(d)

$$\bar{F} = \begin{pmatrix} -d & \frac{-3}{2d^2} \\ 0 & \frac{-3d}{2} \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} 2d^2 & 0 \\ -2d^3 & -2 \end{pmatrix}$$

Now

$$N = \begin{pmatrix} 2d^2 & 0 & -2d^3 & -3 \\ -2d^3 & -2 & 2d^4 & 6d \end{pmatrix}$$

and via the procedure given above, we find that $\gamma_1 = \begin{pmatrix} -1 \\ 2d \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

generate the columns of N over R[d].

Hence,

$$T = J^{-1}\,(\gamma_1, \gamma_2) = \begin{pmatrix} \frac{-1}{2d^2} & 0 \\ -d/2 & 1 \end{pmatrix}$$

Then,

$$F = T^{-1}\bar{F}T = \begin{pmatrix} -d & 3 \\ 0 & 0 \end{pmatrix}, \quad G = T^{-1}\bar{G} = \begin{pmatrix} -d^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $H = \bar{H}T = \begin{pmatrix} -2 & 0 \\ 2d & -2 \end{pmatrix}$.

Since

$$\frac{1}{\det\ (pI-F)} = \frac{1}{p^2+pd} \equiv h(t+1) - \sum_{n=o}^{\infty} \frac{(n-t-1)^n}{n!} h(t+1-n)$$

which belongs to V, the decomposition $(F,G,H)_n$ over R[d] yields a realization of minimal size of the input/output operator f. In component form the realization is given by

$$\frac{dx_1(t)}{dt} = -x_1(t-1) + 3x_2 - u_1(t-2)$$

$$\frac{dx_2(t)}{dt} = u_2(t)$$

$$y_1(t) = -2x_1(t)$$

$$y_2(t) = 2x_1(t-1) - 2x_2(t)$$

References

1. L. Schwartz, Theorie des Distributions, Hermann, Paris, 1966.

2. O. Zariski and P. Samuel, Commutative Algebra, Volume 1, Van Nostrand, Princeton, 1958.

3. H. Blomberg, J. Sinervo, A. Halme, and R. Ylinen, "On Algebraic Methods in Systems Theory", ACTA Poly. Scandinavica, Series #19, Helsinki, 1969.

4. S. MacLane and G. Birkhoff, Algebra, Macmillan, 1967.

5. S. Lang, Algebra, Addison-Wesley, 1965.

6. R. Bellman and K. Cooke, Differential-Difference Equations, Academic Press, 1963.

7. M. Oğuztöreli, Time-Lag Control Systems, Academic Press, 1966.

8. R. Kalman, P. Falb, and M. Arbib, Topics in Mathematical System Theory, McGraw-Hill, 1969.

9. E. Kamen, "A Distributional-Module Theoretic Representation of Linear Continuous-Time Systems", Rept. SEL-71-044 (TR.No. 6560-24), Stanford Electronics Labs, Stanford, Calif., 1971.

10. B. Ho, "On Effective Construction of Realizations from Input-Output Description", Ph.D. Dissertation, Stanford Univ., 1966.

11. R. Newcomb, "On the Realization of Multivariable Transfer Functions", Research Report EERL 58, Cornell Univ., 1966.

12. Y. Rouchaleau, "Linear, Discrete Time, Finite Dimensional, Dynamical Systems Over Some Classes of Commutative Rings", Ph.D. Dissertation, Stanford Univ., 1972.

13. Y. Rouchaleau, B. Wyman, and R. Kalman, "Algebraic Structure of Linear Dynamical Systems. III. Realization Theory Over a Commutative Ring", Proc. Natl. Aca. Science, Vol. 69, No. 11, pp. 3404-3406, Nov. 1972.

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NEW APPROACHES TO LINEAR SYSTEMS

Final Report

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by

Edward W. Kamen
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

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SUMMARY

A new theory of linear systems is given in terms of operator rings rather than linear spaces over scalar fields. New results and computational procedures are presented for large classes of infinite-dimensional continuous-time systems and time-varying discrete-time systems. Applications of the operator theory to problems in realization, control, etc., are considered. The work consists of the following two parts.

Part I. For a large class of linear continuous-time systems, including delay-differential systems, an algebraic theory is presented in terms of Noetherian operator rings generated from a finite number of elements belonging to a convolution algebra of distributions. The external behavior of these systems is given by a finite set of input/output (convolution) operator equations which are solved in a novel manner by constructing an operational transfer function matrix and then applying an extension of the Mikusinski operational calculus. After the formulation of an internal representation consisting of a finite set of scalar operational-differential equations, the problem of realizing an operational transfer function matrix by such an internal description is considered. Results on the existence and construction of realizations are given.

Part II. A theory of linear time-varying discrete-time systems is constructed in terms of a variable time reference which yields a new type of global-in-time representation. In this approach, the time-variance of systems is incorporated into an algebraic framework consisting of modules defined over noncommutative rings. In particular, input/output behavior is specified by a homomorphism between modules over a noncommutative ring

of formal power series, yielding an operational calculus for computing system responses. Dynamical behavior is given in terms of a module structure defined over a skew polynomial ring. This framework is utilized to obtain general results on reachability and controllability, and is then applied to the problem of realizing time-varying discrete-time systems.

APPLICATIONS OF THE RESEARCH

Part I of the research can be applied to important types of operational-differential systems, including systems with time delays, such as biological systems, communication systems, and energy transmission systems. In particular, a new algebraic procedure, having advantages over the usual numerical methods, has been developed for the analysis of linear time-invariant operational-differential systems. The results of this research can also be utilized to construct state-type models of operational-differential systems.(the problem of realization or synthesis).

Part II of the work deals with a new algebraic approach to time-varying discrete-time systems, such as sampled-data systems and various types of sequential systems with switching operations. Potential applications include the development of procedures for the synthesis of optimal linear digital filters and methods for the design of adaptive control systems.

RESULTS

A complete description of the research is contained in the following two papers, "On an Algebraic Theory of Systems Defined by Convolution Operators" and "A New Algebraic Approach to Linear Time-Varying Systems." The first paper has been accepted for publication in Mathematical Systems Theory. The second paper has been submitted for publication. This research was carried out by Dr. Edward W. Kamen, Assistant Professor of Electrical Engineering, Georgia Institute of Technology.

ON AN ALGEBRAIC THEORY OF SYSTEMS DEFINED BY CONVOLUTION OPERATORS*

E. W. Kamen

School of Electrical Engineering

Georgia Institute of Technology

Atlanta, Georgia

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ABSTRACT

For a large class of linear continuous-time systems including delay-differential systems, an algebraic theory is presented in terms of Noetherian operator rings generated from a finite number of elements belonging to a convolution algebra of distributions. The external behavior of these systems is given by a finite set of input/output (convolution) operator equations which are solved in a novel manner by constructing an operational transfer function matrix and then applying an extension of the Mikusiński operational calculus. After the formulation of an internal representation consisting of a finite set of scalar operational-differential equations, the problem of realizing an operational transfer function matrix by such an internal description is considered. Results on the existence and construction of realizations are given.

I. Introduction

The practicality of existing theories on finite-dimensional time-invariant systems is mainly a result of "finiteness" in the structural aspects of the mathematical representation; for example, the finite degree of polynomials in the transfer function matrix or the finite size of matrices in the classical state space description. Unfortunately, these nice characteristics are lost when the theory is extended to include infinite-dimensional systems. As a result, in the infinite-dimensional case the usual transform and state space techniques do not yield practical computational procedures unless an approximation theory is implemented.

However, in many cases the infinite elements (or "devices") within a system play an integral role in system behavior such that approximations (by lumped elements, say) can not be made without losing the direct relationships between system properties and the characteristics of the infinite elements. A very common example is a continuous-time system containing ideal time delays with important system properties depending on the magnitudes of the time delays.

In the study of infinite-dimensional systems there is a need for an algebraic theory whose structural properties are "finite" so that computations within this framework can be performed. Algebraic techniques have been applied to various special situations, but a sufficiently general algebraic framework is still lacking even in the time-invariant case. The purpose of this paper is to establish the foundations of such a theory for a large class of infinite-dimensional time-invariant continuous-time systems including delay-differential systems.

The basic idea of the approach given here is to construct a one-to-one correspondence between the "primitive" elements of a system and the generators of the mathematical representation of the system. More precisely, given a class of systems consisting of an interconnection of elements having impulse responses $\theta_1, \theta_2, \ldots, \theta_q$ belonging to a convolution algebra V of distributions, the systems in this class are represented by (convolution) operators belonging to the subring of V generated by $\theta_1, \ldots, \theta_q$. The study of the input/output representation in terms of these operators is given in Sections 2,3, and 4.

The finiteness of this algebraic framework is primarily a result of the fact that the ring of operators is a Noetherian domain which means that every ideal is finitely generated. Furthermore, in many cases the elements $\theta_1, \ldots, \theta_q$ are algebraically independent (as defined in Section 2) with the result that the operator ring is isomorphic to the polynomial ring in q symbols. In these cases the rich theory of polynomials in several symbols can be utilized to study the given class of systems. In Sections 5 and 6, it is shown that the description of operational-differential systems in terms of polynomials over Noetherian domains makes it possible to compute internal representations from an operational transfer function matrix. An example of the realization procedure is given in Section 7.

2. Input/Output Description

Let R denote the field of real numbers and let V denote the linear space of R-valued Schwartz distributions (generalized functions) defined on R with supports bounded on the left. As proved by Schwartz [1], with addition (+) and convolution (*) V is also a commutative ring with no divisors of zero. (In other words, V is an integral domain.) Further, the linear structure on V is compatible with the ring structure in that V is a convolution algebra over R. The identity of V is the Dirac distribution δ_o. Note that R can be viewed as a subring of V under the embedding $R \to V: a \mapsto a\delta_o$.

Letting U denote a fixed linear subspace of V, we shall be concerned mainly with systems consisting of an interconnection of devices having inputs belonging to U and outputs belonging to V. In particular, given a single-input single-output device, the input/output behavior is specified by the linear operator $U \to V : u \mapsto \theta * u$ where $\theta \in V$ is the impulse response of the device. Common examples of such devices are integrators (θ = Heaviside function) and scalors for which $\theta = a\delta_o$, $a \in R$.

As is well known, operators of the form $u \mapsto \theta * u$, $\theta \in V$, can also represent devices that are distributed in an axial direction. For example, $\theta = \delta_a$, $a > 0$, can be viewed as the impulse response of an LC transmission line (or ideal delay line) with time delay a. Elements of V are also utilized to specify lossy and dispersive delay lines of various types. An example of the latter is an RC transmission line with
$\theta = .5(\pi t^3)^{-\frac{1}{2}}$ a exp $(-a^2/4t)$ h(t), h(t) = Heaviside function.

Generalizing, it is true that many devices of practical interest can be specified by an input/output operator $U \to V : u \mapsto \theta * u$ with $\theta \in V$. However, V may be unnecessarily "large" for some applications, in which case a theory corresponding to that given below could be constructed in terms of a proper subalgebra of V or some other convolution algebra. Also we could consider convolution algebras defined over the field of complex numbers.

The systems under consideration here are described in terms of (convolution) operator equations that are generated in the following manner. Given a finite list of fixed elements $\theta_1, \theta_2, \ldots, \theta_q$ belonging to V, let $R[\theta_1, \ldots, \theta_q]$ denote the smallest subring of V containing $\theta_1, \ldots, \theta_q$ and R (viewed as a subring of V). An element $\alpha(\theta_1,\ldots,\theta_q)$ in $R[\theta_1,\ldots,\theta_q]$ can be written as a finite sum

$$\alpha(\theta_1, \ldots \theta_q) = \sum_{j_1,\ldots,j_q} a_{j_1,\ldots,j_q} \theta_1^{j_1} * \theta_2^{j_2} * \ldots * \theta_q^{j_q}$$

where the j_i are non-negative integers, $a_{j_1,\ldots,j_q} \in R$, $\theta_i^j = j^{th}$ fold convolution of θ_i, $\theta_i^o = \delta_o$. The ring $R[\theta_1,\ldots,\theta_q]$ is an integral domain since it is a subring of the integral domain V.

Any fixed $\alpha(\theta) \in R[\theta_1,\ldots,\theta_q]$ (θ denotes the list $\theta_1, \ldots, \theta_q$) defines a linear operator $V \to V: v \mapsto \alpha(\theta)*v$. With the usual addition and composition, the set of all operators on V of the form $v \mapsto \alpha(\theta)*v$, $\alpha(\theta) \in R[\theta_1,\ldots,\theta_q]$, is a ring which is isomorphic to $R[\theta_1,\ldots,\theta_q]$. For this reason, we shall usually refer to $R[\theta_1,\ldots,\theta_q]$ as a ring of operators.

For any fixed operator ring $R[\theta_1,\ldots,\theta_q]$, we consider the class of m-input terminal k-output terminal systems whose external description is given by the following finite set of (convolution) operator equations

(1) $$\sum_{j=1}^{k} \alpha_{ij}(\theta)*y_j = \sum_{j=1}^{m} \beta_{ij}(\theta)*u_j, \; i = 1, 2, \ldots, k$$

where $\alpha_{ij}(\theta)$, $\beta_{ij}(\theta) \in R[\theta_1,\ldots,\theta_q]$, the $y_j \in V$ are the outputs, and the $u_j \in U$ are the inputs, U = fixed linear subspace of V.

If $q = 1$ and $\theta = \delta_o^{(1)} =$ first derivative of δ_o, then since

$\left(\delta_o^{(1)}\right)^n *v = \delta_o^{(n)} *v = n^{th}$ derivative of $v \in V$,

(1) is a set of ordinary linear constant differential equations which is often taken as the input/output representation of a finite-dimensional time-invariant system. If $q = 1$ and θ is any fixed element of V, then (1) could be the external representation of a system consisting of an interconnection of a finite number of adders, scalors, and devices having impulse response θ (or θ^{-1} if θ is invertible in V). More generally, if $\theta = \theta_1,\ldots,\theta_q$, then (1) could represent a system consisting of an interconnection of adders, scalors, and finite combinations of devices having

impulse responses $\theta_1,\ldots,\theta_q$ (or θ_i^{-1} if θ_i is invertible in V). This latter case includes a large class of infinite-dimensional systems which are defined in terms of the following notions.

A device with impulse response $\theta \in V$ is said to be finite (or lumped) if there exist elements α, $\beta \in R[p]$, $p = \delta_o^{(1)}$, such that $\beta * \theta = \alpha$. A device is infinite if it is not finite. Via standard constructions in realization theory, it can be shown that a device admits a finite-dimensional state space representation if and only if it is finite, hence the motivation for the term finite. Integrators and scalors are two common examples of finite devices. Examples of infinite devices are ideal delay lines and dispersive delay lines.

If we then let $\theta = p,\theta_1,\ldots,\theta_r$ where $p = \delta_o^{(1)}$, the set of operator equations (1) could represent a system consisting of an interconnection of finite devices and finite combinations of infinite devices having impulse responses $\theta_1,\ldots,\theta_r$ (or θ_i^{-1}). These systems will be referred to as operational-differential systems. Common examples are delay-differential systems in which the infinite devices are ideal delay lines.

An obvious but important point is that the properties of a system specified by (1) depend on the algebraic properties of the operator ring $R[\theta_1,\ldots,\theta_q]$. To determine the structure of this ring, in the remainder of this section we relate it to the ring of polynomials over R in q symbols.

Let $R[s_1, s_2, \ldots, s_q]$ denote the ring of polynomials in the symbols $s_1, s_2, \ldots, s_q$ with coefficients in R, and define the map

$$\rho : R[s_1,\ldots,s_q] \to R[\theta_1,\ldots,\theta_q] : \alpha(s) \mapsto \alpha(\theta)$$

The map ρ is a surjective ring homomorphism, and thus $R[\theta_1,\ldots,\theta_q]$ is isomorphic to the factor ring $R[s_1,\ldots,s_q]/\ker \rho$ where $\ker \rho = \{\alpha(s) : \rho(\alpha(s)) = 0\}$. Then since $R[\theta_1,\ldots,\theta_q]$ is a homomorphic image of the ring $R[s_1,\ldots,s_q]$ which is Noetherian (by the Hilbert Basis Theorem), it follows that $R[\theta_1,\ldots,\theta_q]$ is also a Noetherian ring. Summing up these results, we have

<u>Proposition 1</u>: Given any finite list $\theta_1,\ldots,\theta_q$ of elements belonging to V, the operator ring $R[\theta_1,\ldots,\theta_q]$ is a Noetherian (integral) domain which is isomorphic to $R[s_1,\ldots,s_q]/\ker\rho$.

The elements $\theta_1,\ldots,\theta_q$ are said to be algebraically independent over R (viewed as a subring of V) if the map $\alpha(s) \mapsto \alpha(\theta)$ is an isomorphism in which case $R[\theta_1,\ldots,\theta_q]$ is isomorphic to the polynomial ring $R[s_1,\ldots,s_q]$. Hence, $\theta_1,\ldots,\theta_q$ are algebraically independent over R if and only if there does not exist a nonzero polynomial $\alpha(s)$ such that $\alpha(\theta) = 0$. For $q = 1$, an element $\theta \in V$, algebraically independent over R, is said to be transcendental over R. Examples of transcendental elements are given in the following

<u>Proposition 2</u>: For any $a \in R$, $a \neq 0$, δ_a is transcendental over R, and $p = \delta_o^{(1)}$ is transcendental over R.

<u>Proof</u>: For $a \neq 0$, δ_a is transcendental over R since the supports of $\delta_a^o = \delta_o$, δ_a, $\delta_a^2 = \delta_{2a},\ldots,\delta_a^n = \delta_{na}$ do not intersect for any positive integer integer n. Let $\alpha(s) = \sum_{i=0}^{n} a_i s^i \neq 0$, $a_n \neq 0$, then

$$\alpha(p)*h^n = a_n\delta_o + \sum_{i=0}^{n-1} a_i h^{n-i} \neq 0,\ h = \text{Heaviside function, since}$$

$$\operatorname{supp}\left(\sum_{i=0}^{n-1} a_i h^{n-i}\right) \neq \{0\}\ .$$

Examples of algebraically independent elements are given in the following

†Theorem 1: Given $a_1,\ldots,a_r \in R$, the elements $p, \delta_{a_1},\ldots,\delta_{a_r}$ are algebraically independent over R if and only if given $m_1,\ldots,m_r \in Z$ = integers, such that $m_1a_1+\ldots+m_ra_r = 0$, then $m_i = 0$, all i.

Proof: The necessity of the condition on the a_i is clear, for suppose that there exist $m_i \in Z$ with $m_i \neq 0$ for at least one i, such that $\sum m_ia_i = 0$. Then $\delta_{a_1}^{m_1} * \ldots * \delta_{a_r}^{m_r} - \delta_o = 0$ which shows that $\delta_{a_1},\ldots,\delta_{a_r}$ are not algebraically independent. To prove sufficiency we shall use the following result from ring theory.

Lemma: $\theta_1,\ldots,\theta_q$, $q > 1$, are algebraically independent over R if and only if each θ_i, $i = 1,2,\ldots,q$, is transcendental over $R[\theta_1,\ldots,\theta_{i-1}]$.

Proof of Sufficiency in Theorem 1: When $r = 0$, p is transcendental over R by Proposition 2. Now let

$$0 \neq \alpha(s) = \pi_n s^n+\ldots+\pi_1 s + \pi_o \in R[p,\delta_{a_1},\ldots,\delta_{a_{r-1}}][s],\ r \geq 1,$$

and let $a_1,\ldots,a_r$ satisfy the hypothesis of the theorem. Then $\alpha\left(\delta_{a_r}\right) \neq 0$ since the supports of the $\pi_i\delta_{a_r}^i$ do not intersect for all i. Thus δ_{a_r} is transcendental over $R[p,\delta_{a_1},\ldots,\delta_{a_{r-1}}]$, and by the above lemma the proof of the theorem is complete.

In many cases of interest, the elements $\theta_1,\theta_2,\ldots,\theta_q$ generating the operator ring $R[\theta_1,\ldots,\theta_q]$ are algebraically independent over R. For example, it can be shown that any delay-differential system can be specified by the set of equations (1) where $\theta = p, \delta_{a_1},\ldots,\delta_{a_r}$ with the a_i as in the above theorem.

3. Quotient Field Operations

Given the finite set of equations (1), in this section we consider the

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† The condition on the a_i was given by one of the reviewers.

existence and construction of solutions by utilizing operations in quotient fields.

Since the ring V of distributions is an integral domain, the smallest field in which V can be embedded is its quotient field, denoted by Q. The elements of Q are equivalence classes whose representatives are denoted by $\frac{u}{v}$ where $u, v \in V, v \neq 0$. The ring V is embedded in its quotient field by the map $\mathcal{J}: V \to Q: v \mapsto \frac{v}{\delta_o}$. Usually, $\mathcal{J}(v)$ will be denoted by v.

Now let $\theta_1, \theta_2, \ldots, \theta_q$ be a finite list of fixed elements in V as before. Then since the ring $R[\theta_1, \ldots, \theta_q]$ is an integral domain, the smallest field in which $R[\theta_1, \ldots, \theta_q]$ can be embedded is its quotient field, denoted by $R(\theta_1, \ldots, \theta_q)$. Clearly, $R(\theta_1, \ldots, \theta_q)$ is a subfield of Q, and in fact it is the smallest subfield of Q containing $\theta_1, \ldots, \theta_q$ and R (viewed as a subfield of Q).

For positive integers m and k, let $R[\theta_1, \ldots, \theta_q]^{k \times m}$ denote the $R[\theta_1, \ldots, \theta_q]$-module of $k \times m$ matrices over $R[\theta_1, \ldots, \theta_q]$, and let V^m denote the free V-module of m-element column vectors over V. With respect to this notation, the set of equations (1) can be written in the following matrix form:

$$(2) \qquad A(\theta) * y = B(\theta) * u$$

where $A(\theta) = (\alpha_{ij}(\theta)) \in R[\theta_1, \ldots, \theta_q]^{k \times k}$,

$B(\theta) = (\beta_{ij}(\theta)) \in R[\theta_1, \ldots, \theta_q]^{k \times m}$, $y = (y_1, \ldots, y_k)^{TR} \in V^k$,

and $u = (u_1, \ldots, u_m)^{TR} \in U^m$ (TR = transpose).

Let $R(\theta_1, \ldots \theta_q)^{k \times k}$ denote the ring of $k \times k$ matrices over the quotient field $R(\theta_1, \ldots, \theta_q)$. By a well-known result in matrix algebra, $A(\theta) \in R[\theta_1, \ldots, \theta_q]^{k \times k}$ has a (unique) inverse in the matrix ring $R(\theta_1, \ldots, \theta_q)^{k \times k}$ if and only if the determinant of $A(\theta)$, denoted by $\det A(\theta)$, is not zero. If $\det A(\theta) \neq 0$, we denote the inverse of $A(\theta)$ by $A(\theta)^{-1}$. Finally, letting $V^{k \times k}$ denote the V-module of $k \times k$ matrices over V, we have the following results on the existence of solutions of (2).

Proposition 3: If $\det A(\theta) \neq 0$, for any $u \in V^m$ (2) has the unique solution $y = A(\theta)^{-1} \cdot B(\theta) * u \in V^k$ if and only if $A(\theta)^{-1} \cdot B(\theta) \in V^{k \times m}$, where $\cdot$ denotes that componentwise multiplications are in Q.

Proof: If : Viewing (2) as a set of equations over Q, we obtain the solution $y = A(\theta)^{-1} \cdot B(\theta) \cdot u \in Q^k$. Then since $A(\theta)^{-1} \cdot B(\theta) \in V^{k \times m}$, $y = A(\theta)^{-1} \cdot B(\theta) \cdot u \in V^k$, all $u \in V^m$.

Only if: Let $W(\theta) = (w_{ij}(\theta)) = A(\theta)^{-1} \cdot B(\theta)$, and for every $j = 1,2,\ldots,m$, let e_j be the element of V^m all of whose components are zero except for the j^{th} which is equal to δ_o.

Then since $y = W(\theta) \cdot e_j = (w_{1j},\ldots,w_{kj})^{TR} \in V^k$ for all j, we have that $W(\theta) \in V^{k \times m}$.

Corollary 1: If $\det A(\theta)$ has an inverse $(\det A(\theta))^{-1} \in V$, then for any $u \in V^m$, (2) has a unique solution y in V^k, given by

$$y = (\det A(\theta))^{-1} * \tilde{A}(\theta) * B(\theta) * u \tag{3}$$

where $\tilde{A}(\theta)$ is the transpose of the matrix of cofactors of $A(\theta)$.

Proof: Let $V^{k \times k}$ denote the ring of $k \times k$ matrices over the ring V. Then $R[\theta_1,\ldots,\theta_q]^{k \times k}$ is a subring of $V^{k \times k}$ and by standard results in matrix algebra, $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k \times k}$ has a (unique) inverse in $V^{k \times k}$ if and only if $\det A(\theta)$ is a unit (invertible element) in V. If $\det A(\theta)$ is a unit in V, then $A(\theta)^{-1} = (\det A(\theta))^{-1} * \tilde{A}(\theta)$ and $A(\theta)^{-1} * B(\theta) \in V^{k \times m}$. Hence, by the above propositon, the corollary is proved.

Corollary 2: If $B(\theta)$ has an $m \times k$ right inverse over V, then for any $u \in V^m$ (2) has a unique solution $y \in V^k$ if and only if $\det A(\theta)$ is a unit in V.

Proof: The if part follows from Proposition 3. Now suppose that (2) has a unique solution $y \in V^k$ for any $u \in V^m$, and for every $i = 1,2,\ldots,k$, let w_{ij} denote the i^{th} component of the solution when $u = e_j$, $j = 1,2,\ldots,m$. Then $A(\theta)*W = B(\theta)$ where $W = (w_{ij})$, and if $B(\theta)$ has a right inverse M over V, we have that $A(\theta)*(W*M) = I_k$ where I_k is the $k\times k$ identity matrix. Thus, $A(\theta)$ is a unit in $V^{k\times k}$ which implies that det $A(\theta)$ is a unit in V.

Note that if $A(\theta)^{-1}\cdot B(\theta) \in V^{k\times m}$, the system given by (2) can be represented by the restriction to U^m of the input/output operator $f:V^m \to V^k: v \mapsto A(\theta)^{-1}\cdot B(\theta)*v$. With respect to the free V-module structure on V^m and V^k, f is a V-module homomorphism and $A(\theta)^{-1}\cdot B(\theta)$ can be viewed as the matrix representation of f relative to the standard bases in V^m and V^k.

As seen from (3), solutions of (2) can be computed by first finding the inverse in V of det $A(\theta)$ if it exists. To simplify this computation, we can utilize the property that the operator ring $R[\theta_1,\ldots,\theta_q]$ is Noetherian. As a consequence, every nonunit of the ring $R[\theta_1,\ldots,\theta_q]$ can be written as a finite product of factors that are irreducible in $R[\theta_1,\ldots,\theta_q]$. Further if $\theta_1,\ldots,\theta_q$ are algebraically independent over R, then $R[\theta_1,\ldots,\theta_q]$ is a unique factorization domain, and hence factorizations into irreducible elements are unique. Now as a consequence of the commutative ring structure on V, we have the following

Lemma: Let det $A(\theta)$ be a nonunit in $R[\theta_1,\ldots,\theta_q]$ and let det $A(\theta) = \pi_1*\pi_2*\ldots*\pi_\ell$ be a decomposition of det $A(\theta)$ into irreducible factors. Then det $A(\theta)$ is a unit in V if and only if each π_i has an inverse π_i^{-1} in V, in which case $(\det A(\theta))^{-1} = \pi_1^{-1}*\pi_2^{-1}*\ldots*\pi_\ell^{-1}$.

Combining Corollary 1 of Proposition 3 and the above lemma, we have

<u>Theorem 2</u>: If $\det A(\theta)$ is a non-unit in $R[\theta_1,\ldots,\theta_q]$ and if $\det A(\theta) = \pi_1 * \pi_2 * \ldots * \pi_\ell$ is a decomposition of $\det A(\theta)$ into irreducible factors with each π_i having inverse π_i^{-1} in V, then for any $u \in V^m$, (2) has the unique solution

$$y = (\pi_1^{-1} * \pi_2^{-1} * \ldots * \pi_\ell^{-1}) * \tilde{A}(\theta) * B(\theta) * u \in V^k \tag{4}$$

A fundamental point here is that in determining the existence of solutions of (2) assuming $\det A(\theta) \neq 0$, it is sufficient to consider the invertibility in V of the irreducible elements of the operator ring $R[\theta_1,\ldots,\theta_q]$. In particular, combining the above results, we obtain:

<u>Theorem 3</u>: If every irreducible element of $R[\theta_1,\ldots,\theta_q]$ is a unit in V, then the quotient field $R(\theta_1,\ldots,\theta_q)$ is contained in V, and for any $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$ and $B(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times m}$ with $\det A(\theta) \neq 0$, (2) has the unique solution (3) for all $u \in V^m$.

As an application of Theorem 3, let $q = 1$ and let $\theta \in V$ be any fixed element transcendertal over R. Then the only irreducible elements of $R[\theta]$ are linear elements $a\theta + b$ and quadratic elements $a\theta^2 + b\theta + c$ with negative discriminant $b^2 - 4ac < 0$. Hence if these elements are units in V, we have that $R(\theta) \subset V$. For example, it is easy to show (and already known) that these elements are units in V when $\theta = p$ or $\theta = \delta_a$, and thus both $R(p)$ and $R(\delta_a)$ are contained in V.

When $\theta = \theta_1,\ldots,\theta_q$, $q > 1$, the problem of determining the irreducible elements of $R[\theta_1,\ldots,\theta_q]$ is very difficult in general. However, when θ_i is transcendental over $R[\theta - \theta_i]$ for some fixed i, where $\theta - \theta_i$ denotes the list $\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_q$, we could consider a splitting field over $R(\theta - \theta_i)$. We leave this for further studies.

An example in which $R(\theta_1,\ldots,\theta_q)$, $q > 1$, is contained in V is given in the following

<u>Proposition 4</u>: For any $a_1,\ldots,a_r \in R$, the field $R(p,\delta_{a_1},\ldots,\delta_{a_r})$ is contained in V.

<u>Proof</u>: Let $0 \neq \pi \in R[p,\delta_{a_1},\ldots,\delta_{a_r}]$ and consider π^{-1} which belongs to $R(p,\delta_{a_1},\ldots,\delta_{a_r})$. It will be shown that $\pi^{-1} \in V$. Multiply π^{-1} by $\frac{\sigma}{\sigma}$ with σ consisting of Dirac distributions so that $\pi^{-1} \in R(p,\delta_{\bar{a}_1},\ldots,\delta_{\bar{a}_r})$, where $\bar{a}_i \geq 0$ for all i.

Now $\pi^{-1} \in R(p)(\delta_{\bar{a}_1},\ldots,\delta_{\bar{a}_r})$ and we can expand this into a formal series in the elements $\delta_{\bar{a}_1},\ldots,\delta_{\bar{a}_r}$ with coefficients in R(p); that is

$$\pi^{-1} = \sum_{j_1,\ldots,j_r}^{\infty} \pi_{j_1,\ldots,j_r}\, \delta_{\bar{a}_1}^{j_1} * \cdots * \delta_{\bar{a}_r}^{j_r}$$

where $j_i \geq N_i$ with $-\infty < N_i \leq 0$. Reorder the terms of this sum so that

$$\pi^{-1} = \sum_{n\geq 0}^{\infty} \pi_n \delta_{\bar{a}_1}^{j_1} * \cdots * \delta_{\bar{a}_r}^{j_r}, \quad n \sim (j_1,\ldots,j_r)$$

and $(j_1',\ldots,j_r') \sim n' < n \sim (j_1,\ldots,j_r) \Leftrightarrow \sum_{i=1}^{r} \bar{a}_i j_i' \leq \sum_{i=1}^{r} \bar{a}_i j_i$.

Let $\{A_n\}$ denote the sequence of partial sums obtained from this series. For any $\varphi \in \mathcal{D}$ = Schwartz space of infinitely continuously differentiable functions $R \to R$ with compact support, consider the sequence $\{A_n(\varphi)\}$. Since for any $\lambda \in R(p)$ the support of λ as an element of V is contained in $[0,\infty)$, and since $\delta_{\bar{a}_i}^{j_i}$ is concentrated at the point $\bar{a}_i j_i$, only a finite number of the $A_n(\varphi)$ are nonzero. Hence $\{A_n(\varphi)\}$ converges in R, and thus π has an inverse in V.

Note that if $R(\theta_1,\ldots,\theta_q) \subset V$, then V is a linear space over $R(\theta_1,\ldots,\theta_q)$ with the multiplication $R(\theta_1,\ldots,\theta_q)\times V \to V: \left(\frac{\alpha}{\beta},v\right) \mapsto \beta^{-1}*\alpha*v$. In this case, to solve (2) we first can simplify the set of equations by applying the process of Gauss elimination. For the case q = 1 and θ = derivative operator, this approach is similar to that given by Blomberg et al [2].

Unfortunately, even if det $A(\theta)$ can be decomposed into irreducible factors, the actual computation of solutions via (4) is usually quite difficult as a result of the complexity of convolution operations. To simplify the problem of computation, in the next section we present an algebraic procedure that extends the operational calculus of Mikusiński [3].

4. Computation of Solutions

Given a system specified by the input/output equations

$$A(\theta)*y = B(\theta)*u \tag{5}$$

where $A(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times k}$ with det $A(\theta) \neq 0$, and $B(\theta) \in R[\theta_1,\ldots,\theta_q]^{k\times m}$, the expression of $A(\theta)^{-1}\cdot B(\theta)$ as an element of $R(\theta_1,\ldots,\theta_q)^{k\times m}$ (i.e., as a matrix of rational functions) is referred to as the operational transfer function matrix of the system. In correspondence with the standard terminology, the expression of $A(\theta)^{-1}\cdot B(\theta)$ as an element of $V^{k\times m}$ (assuming that $A(\theta)^{-1}\cdot B(\theta)$ is contained in $V^{k\times m}$) is called the impulse response function matrix of the system.

Since $R(\theta_1,\ldots,\theta_q)$ is a field, the inverse of $A(\theta)$ over $R(\theta_1,\ldots,\theta_q)$ can be computed by the usual techniques of inverting matrices over a field. Hence the operational transfer function matrix $A(\theta)^{-1}\cdot B(\theta)$ can be determined from (5) by using standard procedures. Furthermore, if the input u can be expressed as an element of $R(\theta_1,\ldots,\theta_q)^m$, then as an element of $R(\theta_1,\ldots,\theta_q)^k$, the output y is readily computed. The main difficulty in obtaining solutions of (5) via this procedure is expressing y as an element of V^k (when possible). Hence the central problem is computing the inverse image, under the embedding $\mathcal{S}:V \to Q \supset R(\theta_1,\ldots,\theta_q)$, of elements in $R(\theta_1,\ldots,\theta_q)$. We now consider an algebraic method which simplifies this problem.

First, let $q = 1$ and let $\theta \in V$ be any fixed transcendental element over R.

Since $R[\theta]$ is a principal ideal domain (pid) by applying the method of partial fraction expansions, we can reduce the problem of finding the inverse image under $\mathcal{J}$ of elements in $R(\theta)$ to computing the inverses in V of linear and quadratic elements in $R[\theta]$ (or linear elements in $C[\theta]$, C = field of complex numbers). If $\theta = p$ = derivative operator, this procedure yields the operational calculus of Mikusiński [3]. The main point here is that operational techniques apply to any class of systems described by operators in $R[\theta]$ with $\theta \in V$ transcendental over R.

Now let us consider the case when $q > 1$ and $\theta_1,\ldots,\theta_q$ are algebraically independent. For any fixed i, the elements of $R(\theta_1,\ldots,\theta_q)$ can be viewed as elements in the quotient field of the ring $R(\theta-\theta_i)[\theta_i]$. Then since $R(\theta-\theta_i)[\theta_i]$ is a pid, any element of $R(\theta_1,\ldots,\theta_q)$, viewed as an element of $R(\theta-\theta_i)(\theta_i)$, can be decomposed via a <u>repeated</u> partial fraction expansion as follows.

Let Ω denote the set of monic irreducible polynomials in $R(\theta-\theta_i)[\theta_i]$. After multiplication by elements in $R[\theta-\theta_i]$ if necessary, the elements of Ω actually belong to $R[\theta-\theta_i][\theta_i]$ since if α is an irreducible polynomial in $R[\theta-\theta_i][\theta_i]$, it is also an irreducible polynomial in $R(\theta-\theta_i)[\theta_i]$ (see Zariski and Samuel [4, page 102]). Then given any element $\frac{\alpha}{\beta} \in R(\theta_1,\ldots,\theta_q)$, $\frac{\alpha}{\beta}$ has a unique decomposition

$$(6) \qquad \frac{\alpha}{\beta} = \sum_{\omega \in \Omega} \frac{\pi_\omega}{\omega^{j(\omega)}} + \gamma$$

where $\pi_\omega, \gamma \in R(\theta-\theta_i)[\theta_i]$, $j(\omega)$ are non-negative integers, $\pi_\omega = 0$ if $j(\omega)=0$, π_ω is relatively prime to ω if $j(\omega) > 0$, and $\deg \pi_\omega < \deg \omega^{j(\omega)}$ if $j(\omega) > 0$.

<u>Note</u>: The expression (6) can be decomposed further by viewing the π_ω as polynomials in $R(\theta-\theta_j)(\theta_j)$, $i \neq j$, and then applying the partial expansion to each π_ω, and so on.

This procedure can greatly simplify the problem of finding the inverse image of elements in $R(\theta_1,\ldots,\theta_q)$. For illustrative purposes, we present the following

Example: Consider the delay-differential system given by the following input/output equations

$$\frac{d^2y_1(t)}{dt^2} + \frac{dy_1(t-1)}{dt} + y_2(t-1) + y_2(t) = \frac{du_1(t-1)}{dt} + 2u_1(t) + \frac{2du_2(t-2)}{dt}$$

$$\frac{-dy_1(t)}{dt} - y_1(t-1) + y_2(t) = -u_1(t-1) - u_2(t-2) .$$

If we let $d = \delta_1$ and $p = \delta_o^{(1)}$, then the system can be specified in terms of operators belonging to the ring $R[d,p]$ with d and p algebraically independent over R (by Theorem 1).

In matrix form, we have $A(p,d)*y = B(p,d)*u$ where

$$A(p,d) = \begin{pmatrix} p^2+dp & d+1 \\ -p-d & 1 \end{pmatrix} , \quad B(p,d) = \begin{pmatrix} dp+2 & 2d^2p \\ -d & -d^2 \end{pmatrix}$$

Computing the inverse $A(p,d)^{-1}$ of $A(p,d)$ over $R(p,d)$ and multiplying by $B(p,d)$, we obtain

$$A(p,d)^{-1}\cdot B(p,d) = \frac{1}{p^2+(2d+1)p+d^2+d} \begin{pmatrix} dp+d^2+d+2 & 2d^2p+d^3+d^2 \\ 2p+2d & d^2p^2+d^3p \end{pmatrix}$$

which is the operational transfer function of the system (we are omitting the * ~~[illegible]~~ for convolution). Now let $u = \left(-e^{-t}h(t),\ h(t)\right)^{TR}$ where $h(t)$ = Heaviside function. As an element of $R(p,d)^2$, we have that $u = \left(\frac{-1}{p+1}, \frac{1}{p}\right)^{TR}$. Viewing $\det A(p,d) = p^2+(2d+1)p+d^2+d$ as a polynomial in $R(d)[p]$ and applying the quadratic formula, we obtain $\det A(p,d) = (p+d)(p+d+1)$.

Then

$$y = A(p,d)^{-1} \cdot B(p,d) \cdot u = \begin{pmatrix} \dfrac{(2d^2-d)p^2+(d^3+2d^2-d-2)p+d^3+d^2}{p(p+1)(p+d)(p+d+1)} \\ \dfrac{d^2p+d^2-2}{(p+1)(p+d)(p+d+1)} \end{pmatrix}$$

Since det $A(p,d) \neq 0$, by Theorem 3 and Proposition 4, the solution y belongs to V^2. To compute the inverse image under $\mathcal{J}$ of y, we shall view the components of y as elements in the quotient field of $R(d)[p]$ and expand by partial fractions. This gives

(7) $$y_1 = \frac{d}{p} - \frac{\frac{d^2+2}{d(d-1)}}{p+1} + \frac{\frac{d^3-2d^2+2d+2}{d-1}}{p+d} - \frac{\frac{d^3+2}{d}}{p+d+1}$$

(8) $$y_2 = -\frac{\frac{2}{d(d-1)}}{p+1} - \frac{\frac{-d^3+d^2-2}{d-1}}{p+d} + \frac{\frac{-d^3-2}{d}}{p+d+1}$$

Now y_1 and y_2 can be decomposed further by performing the following expansions

(9) $$\frac{d^2+2}{d(d-1)} = 1 - \frac{2}{d} + \frac{3}{d-1}$$

(10) $$\frac{2}{d(d-1)} = \frac{-2}{d} + \frac{2}{d-1}$$

From (7) - (10), it is seen that finding the inverse image of y_1 and y_2 reduces to the problem of inverting p+1, p+d, d-1, and p+d+1. Via power series expansion, we obtain

$$(p+1)^{-1} = e^{-t}h(t)$$

$$(d-1)^{-1} = -\sum_{n=o}^{\infty} \delta_n$$

$$(p+d)^{-1} = \sum_{n=o}^{\infty} \frac{(n-t)^n}{n!} h(t-n) \triangleq f(t)$$

$$(p+d+1)^{-1} = \sum_{n=o}^{\infty} \frac{(n-t)^n}{n!} e^{-(t-n)}h(t-n) \triangleq g(t)$$

Hence, the solution $y = (y_1, y_2)^{TR} \in V^2$ is given by

$$y_1 = h(t-1) - e^{-t}h(t) + 2e^{-(t+1)}h(t+1) + 3\sum_{n=0}^{\infty} e^{-(t-n)}h(t-n)$$

$$-\sum_{n=0}^{\infty}\left[f(t-n-3)-2f(t-n-2)+2f(t-n-1)+2f(t-n)\right]-g(t-2)-g(t+1)$$

$$y_2 = 2e^{-(t+1)}h(t+1)-2\sum_{n=0}^{\infty} e^{-(t-n)}h(t-n) + \sum_{n=0}^{\infty}\left[f(t-n-3)-f(t-n-2)+2f(t-n)\right]$$

$$-g(t-2)-2g(t+1)$$

It is clear from this example that the success of the above algebraic procedure in simplifying the computation of solutions depends on the decomposibility of det $A(\theta)$. When det $A(\theta)$ is decomposable into factors of low degree, this technique of computing solutions compares quite favorably with classical procedures for solving operational-differential equations (see Bellman and Cooke [5]. Furthermore, this approach is an extension of Mikusiński's operational calculus to operator rings in several variables. We also mention that when the generators of the operator ring are specified, the algebraic framework could be applied to equations with initial conditions.

5. Internal Description

Consider a system specified by the input/output operator $f: U^m \to V^k : u \mapsto W*u$ where $W \in V^{k\times m} \cap R(p,\theta_1,\ldots,\theta_r)^{k\times m}$. If the inputs and resulting outputs are regular distributions (generated by locally integrable functions), the classical state space representation of the system (if one exists) is given by

$$\frac{dx(t)}{dt} = Fx(t) + Gu(t)$$

(11)

$$y(t) = Hx(t)$$

where F, G, H are linear maps, $u(t) \in R^m$, $y(t) \in R^k$, and the state $x(t)$ at time t belongs to some locally convex R-linear topological space X, called the state space. If the system contains infinite elements, then X is an infinite-dimensional linear space, and thus the matrix representation of the linear maps F, G, H have infinite size.

To circumvent this infinite dimensionality, we consider the class of "hereditary systems" in which $X = R^n$, $n < \infty$, and the derivative of x(t) at time t depends on x(t) and u(t) over a past interval $[t-\tau, t]$ for some fixed τ, $0 < \tau \leq +\infty$. (Here x(t) is no longer the state in the classical sense.) An example of a finite hereditary system ($\tau < \infty$) is a delay-differential system of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= \sum_{i=1}^{\ell} F_i x(t-b_i) + \sum_{i=1}^{p} G_i u(t-c_i) \\ y(t) &= \sum_{i=1}^{q} H_i x(t-d_i) \end{aligned} \tag{12}$$

where $b_i, c_i, d_i \geq 0$ and F_i, G_i, H_i are matrices over R of size $n\times n$, $n\times m$, and $k\times n$, respectively.

Representations similar to (12) have been used extensively to study the internal properties, such as control, of delay-differential systems. (For example, see Oğuztöreli [6].) Usually, in the literature y(t) = x(t), but it is reasonable to consider situations in which there are also time delays between x(t) and the output y(t).

Our objective here is to extend representations of the form (12) to a general class of operational-differential systems and to do this in terms of operators belonging to the ring $R[p, \theta_1, \ldots, \theta_r]$. First note that we can write (12) in the form

$$\begin{aligned} \frac{dx(t)}{dt} &= (F*x)(t) + (G*u)(t) \\ y(t) &= (H*x)(t) \end{aligned} \tag{13}$$

where F,G,H are matrices of size $n\times n$, $n\times m$, $k\times n$ over the operator ring $R[\delta_{a_1}, \ldots, \delta_{a_r}]$ for some $a_i \in R$. From (13) we obtain the desired generalization as follows

Let $\mathcal{D}$ denote the space of "testing functions" associated with the space of distributions V. Given any positive integer n and $v = (v_1, \ldots, v_n)^{TR} \in V^n$, we define $v(\varphi)$, $\varphi \in \mathcal{D}$, by $v(\varphi) = (v_1(\varphi), \ldots, v_n(\varphi))^{TR} \in R^n$. Given $x \in V^n$, we let $p*x \equiv \frac{dx}{dt}$ denote the operation of $p = \delta_o^{(1)}$ on x in the V-module structure on V^n. We then have the following:

<u>Definition</u>: An m-input terminal k-output terminal operational-differential system over $R[p, \theta_1, \ldots, \theta_r]$ is a triple (F,G,H) of $n \times n$, $n \times m$, $k \times n$ matrices over $R[\theta_1, \ldots, \theta_r]$ such that $(pI-F)^{-1} \cdot G \in V^{n \times m}$, together with the following set of operational-differential equations in the sense of distributions

$$(14) \qquad \begin{aligned} (p*x)(\varphi) \equiv \frac{dx(\varphi)}{dt} &= (F*x)(\varphi) + (G*u)(\varphi) \\ y(\varphi) &= (H*x)(\varphi) \end{aligned}$$

where $u \in U^m$, $y \in V^k$, and $x \in S^n$, S = linear subspace of V. The integer n is called the size of the system.

The set of equations (14) represents a hereditary system in a generalized sense if the components of F,G,H have their supports contained in $[o,\infty)$. Furthermore, (14) is a finite hereditary system if the elements of F,G,H have compact support contained in $[o,\infty)$, which will be the case if the supports of $\theta_1, \ldots, \theta_r$ are compact and contained in $[o,\infty)$. This latter condition implies that the infinite devices comprising the system have impulse responses with compact support $\subset [o,\infty)$, such as ideal delay lines.

Note that by placing suitable constraints on U and F,G,H, we could restrict our attention to operational-differential equations defined in the ordinary sense (that is, we can replace φ by t). We could then consider extending the theory of hereditary systems by using the framework given by (14).

However, the fundamental problem of interest here is constructing representations of the form (14) from operational transfer functions $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m}$.

First, solving (14) over the quotient field Q of V, we have that $x = (pI-F)^{-1}\cdot G*u \in V^n$ since $(pI-F)^{-1}\cdot G \in V^{n\times m}$. Then $y = H\cdot(pI-F)^{-1}\cdot G*u \in V^k$ since H is over $R[\theta_1,\ldots,\theta_r] \subset V$. Hence, $H\cdot(pI-F)^{-1}\cdot G \in R(p,\theta_1,\ldots,\theta_r)^{k\times m} \cap V^{k\times m}$ is the operational transfer function matrix of the system.

We then have the following

Definition of Realization: Given an input/output operator $f:U^m \to V^k : u \mapsto W*u$, $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m} \cap V^{k\times m}$, a realization of f over $R[\theta_1,\ldots,\theta_r]$ is a system of the form (14) with $W = H\cdot(pI-F)^{-1}\cdot G$.

In the next section we pursue the problem of constructing realizations by considering the decomposability of the operational transfer function matrix.

6. Decomposition of Transfer Functions

Again, let $\theta = \theta_1,\ldots,\theta_q$ be a finite list of elements (not necessarily algebraically independent) belonging to V. Given $W \in R(\theta_1,\ldots,\theta_q)^{k\times m}$, we say that W is decomposable over $R(\theta-\theta_i)$ (respectively, over $R[\theta-\theta_i]$) if there exist matrices F,G,H over $R(\theta-\theta_i)$ (respectively $R[\theta-\theta_i]$), where F is $n\times n$, G is $n\times m$, and H is $k\times n$, such that

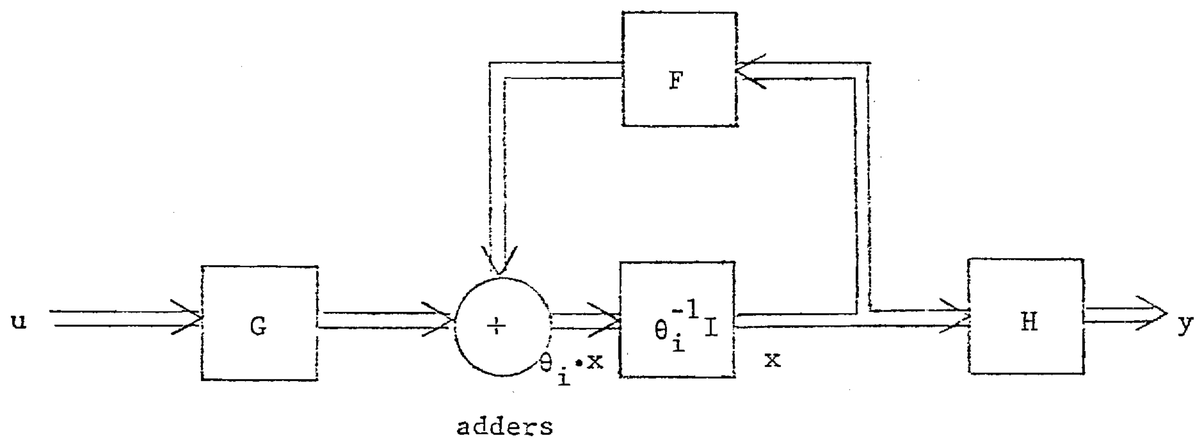
$$W = H\cdot(\theta_i I-F)^{-1}\cdot G. \tag{15}$$

The integer n is called the size of the decomposition. A decomposition $(F,G,H)_n$ over $R(\theta-\theta_i)(R[\theta-\theta_i])$ is said to be minimal if n is minimal among all possible decompositions over $R(\theta-\theta_i)(R[\theta-\theta_i])$.

A decomposition $(F,G,H)_n$ over $R(\theta-\theta_i)$ or $R[\theta-\theta_i]$ yields the following set of operational equations

$$\begin{aligned} \theta_i\cdot x &= F\cdot x + G\cdot u \\ y &= H\cdot x \end{aligned} \tag{16}$$

where $u \in U^m$, and in general $x \in Q^n$, $y \in Q^k$ where Q is the quotient field of V. If θ_i is a unit in V, (16) corresponds to a system with the following "wiring diagram."



This diagram illustrates that in a decomposition over $R(\theta-\theta_i)$ or $R[\theta-\theta_i]$ all devices with impulse response θ_i^{-1} are "extracted." However, in a decomposition over $R(\theta-\theta_i)$ the elements of F,G,H may not be distributions (i.e., they belong to Q) or if they are, their supports may not be contained in $[o,\infty)$ even if the supports of the elements in the list $\theta-\theta_i$ are contained in $[o,\infty)$.

The main interest here in decompositions over $R(\theta-\theta_i)$ is that this problem can be viewed as a first step in determining decompositions over $R[\theta-\theta_i]$ which, in turn, for the case $\theta_i = p = \delta_o^{(1)}$ leads to the construction of realizations as defined in the preceding section. In particular, if $(F,G,H)_n$ is a decomposition of $W \in R(p,\theta_1,\ldots,\theta_r)^{k\times m}$ over $R[\theta-p]$, then $(F,G,H)_n$ defines a realization of the operator $f:U^m \to V^k : u \mapsto W*u$ as given by (14) if $(pI-F)^{-1}\cdot G \in V^{n\times m}$. This latter condition is satisfied if det $(pI-F)$ is a unit in V which is always the case if $R(p,\theta_1,\ldots,\theta_r)$ is contained in V.

Although we are primarily interested in the case $\theta_i = \delta_o^{(1)}$, we consider the construction of decompositions over $R(\theta-\theta_i)$ and $R[\theta-\theta_i]$ for any θ_i such that $R(\theta-\theta_i)[\theta_i]$ is a pid. The main result on decompositions over $R(\theta-\theta_i)$ is given in the following:

Theorem 4: Let θ_i be transcendental over $R[\theta-\theta_i]$ for some fixed i, then $W = \left(\frac{\alpha_{ij}}{\beta_{ij}}\right) \in R(\theta_1,\ldots,\theta_q)^{k\times m}$ is decomposable over $R(\theta-\theta_i)$ if the degree of α_{ij} is less than the degree of β_{ij} for any fixed i,j when α_{ij} and β_{ij} are viewed as elements in $R[\theta-\theta_i][\theta_i]$.

Several constructive proofs of this theorem can be given. The first one that we consider is based on the invariant factor theorem for pids. Let W satisfy the hypothesis of the theorem. Since $R[\theta-\theta_i][\theta_i]$ is contained in $R(\theta-\theta_i)[\theta_i]$, the elements of W can be viewed as elements in the quotient field of the ring $R(\theta-\theta_i)[\theta_i]$ which is a principal ideal domain since θ_i is transcendental over $R[\theta-\theta_i]$. Let ψ be the least common denominator of W as a matrix over $R(\theta-\theta_i)(\theta_i)$. Then since $\deg \alpha_{ij} < \deg \beta_{ij}$, ψW is a matrix over $R(\theta-\theta_i)[\theta_i]$ whose entries have degree less than $\deg \psi$. Since $R(\theta-\theta_i)[\theta_i]$ is a pid, by the invariant factor theorem we can reduce ψW to diagonal form which yields a Smith-McMillan-type form for W. From this the matrices F,G,H of a minimal decomposition can be computed by using Kalman's procedure [7]. For the details along with an example, see Kamen [8].

Other proofs of Theorem 4 are based on a Hankel matrix sequence which is generated in the following manner. Again viewing W as a matrix over the quotient field of the pid $R(\theta-\theta_i)[\theta_i]$, since $\deg \alpha_{ij} < \deg \beta_{ij}$ by long division we can expand W into a formal power series in θ_i^{-1} of the form

$$W = \sum_{\ell=1}^{\infty} A_\ell \theta_i^{-\ell}, \quad A_\ell \in R(\theta-\theta_i)^{k\times m}, \quad \ell = 1,2,3,\ldots$$

We then define a sequence of Hankel matrices for W by

$$\Gamma_{i,j} = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_j \\ A_2 & A_3 & A_4 & \cdots & A_{j+1} \\ A_3 & A_4 & A_5 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ A_i & \cdot & \cdots & & A_{i+j-1} \end{pmatrix}$$

Now if a decomposition of W over $R(\theta-\theta_i)$ exists, then $W = H\cdot(\theta_i I-F)^{-1}\cdot G$. Expanding $(\theta_i I-F)^{-1}$, we obtain

$$H\cdot(\theta_i I-F)^{-1}\cdot G = \sum_{\ell=1}^{\infty} H\cdot F^{\ell-1}\cdot G\cdot\theta_i^{-\ell}.$$

Hence W is decomposable over $R(\theta-\theta_i)$ if there exists matrices F,G,H such that $A_\ell = H\cdot F^{\ell-1}\cdot G$, $\ell = 1,2,3,\ldots$ Again let Ψ = least common denominator of W. Then it follows from the results of Ho [9] that W has a minimal decomposition over $R(\theta-\theta_i)$ of size equal to the rank of $\Gamma_{\sigma,\sigma}$ where σ is the degree of Ψ as an element of $R[\theta-\theta_i][\theta_i]$. The matrices F,G,H can be computed from $\Gamma_{\sigma+1,\sigma+1}$ by using Ho's algorithm. The details have been carried by Newcomb [10] for the case in which W is a rational function in several complex variables.

Another procedure for computing F,G,H is to use Silverman's formulas as derived by Rouchaleau [11]: Let J be a submatrix of $\Gamma_{\sigma,\sigma}$ having maximal rank n. Let K be the $n\times m$ submatrix of the first block column (the first m elementary columns) of $\Gamma_{\sigma,\sigma}$ corresponding to the rows of J. Finally, let L be the $k\times n$ submatrix of the first block row (the first k elementary rows) of $\Gamma_{\sigma,\sigma}$ corresponding to the columns of J. Then $F = J^{-1}M$, $G = J^{-1}K$, $H = L$ is a minimal decomposition where M is the $n\times n$ submatrix of $\Gamma_{\sigma,\sigma+i}$, for i suitably large, sitting to the right of J. An example of this procedure is given in the following section.

Any two minimal decompositions over $R(\theta-\theta_i)$ with θ_i transcendental over $R[\theta-\theta_i]$ are unique in the following sense.

Theorem 5: If $(F,G,H)_n$ and $(\hat{F},\hat{G},\hat{H})_n$ are two minimal decompositions of W over $R(\theta-\theta_i)$, then there exists an $n \times n$ invertible matrix T over $R(\theta-\theta_i)$ such that

$$\hat{F} = TFT^{-1},\ \hat{G} = TG,\ \hat{H} = HT^{-1}.$$

Proof: Let K denote the field $R(\theta-\theta_i)$, write $\Omega = K[\theta_i]$, and let $\Gamma = K[[\theta_i^{-1}]] =$ ring of formal power series over K in θ_i^{-1}. Define $\lambda: \Omega^m \to \Gamma^k : \omega \mapsto W\circ\omega$ where $W\circ\omega$ is the usual multiplication of a matrix of power series in θ_i^{-1} by a vector of polynomials in θ_i with all terms containing nonpositive powers of θ_i^{-1} omitted. Using the constructions given by Kalman [7] in his algebraic theory of discrete-time systems, we have that λ is a K-linear homomorphism and that each minimal decomposition defines a canonical factorization of λ through K^n. The existence of T then follows from Theorem 6.9 and Proposition 6.10 in [7, pages 258 - 259].

Comment: The approach used in the proof of Theorem 5 shows that algebraic results on discrete-time systems can be carried over directly to operational systems in continuous-time, as will be demonstrated again shortly. However, it is very interesting to note that some of the system-theoretic interpretations of the algebraic constructions do not carry over. For example, in the discrete-time theory K^n is the state space, whereas here $K^n = R(\theta-\theta_i)^n$ bears no direct relationship to the state space of the continuous-time system.

We now consider decompositions over $R[\theta-\theta_i]$. Here we utilize the algebraic theory of linear discrete-time systems over commutative rings as developed by Rouchaleau [11] and Rouchaleau, Wyman, and Kalman [12]. One of the main contributions of this work is the application of this algebraic theory to operational systems in continuous-time. In the remainder of this section, we assume that θ_i is transcendental over $R[\theta-\theta_i]$ for some fixed i.

Theorem 6: If W is decomposable over $R(\theta-\theta_i)$ and $W = \sum_{\ell=1}^{\infty} A_\ell \theta_i^{-\ell}$, $A^\ell \in R[\theta-\theta_i]^{k\times m}$, $\ell = 1,2,\ldots$, then W is decomposable over $R[\theta-\theta_i]$.

Proof: Since $R[\theta-\theta_i]$ is a Noetherian domain, it follows directly from the results of Rouchaleau, Wyman, Kalman [12] that W is decomposable over $R[\theta-\theta_i]$. This theorem can also be proved by using the fact that $R[\theta-\theta_i]$ is a Fatou ring as discussed in Cahen and Chabert [13]. Details of this approach in the discrete-time setting are given by Rouchaleau and Wyman [14].

Corollary: If W is decomposable over $R(\theta-\theta_i)$ and W has a common denominator which is monic when viewed as an element of $R[\theta-\theta_i][\theta_i]$, then W is decomposable over $R[\theta-\theta_i]$.

Proof: If W has a monic common denominator, then the A_ℓ in the expansion $W = \sum_{\ell=1}^{\infty} A_\ell \theta_i^{-\ell}$ belong to $R[\theta-\theta_i]^{k\times m}$.

The proof given in [12] of the existence of the matrices F,G,H is fairly constructive. However, in general the decomposition obtained in this manner is not minimal, and as yet there are no practical procedures for computing minimal decompositions over an arbitrary Noetherian ring $R[\theta-\theta_i]$. However, when $q = 2$ and θ_1 and θ_2 are algebraically independent over R, $R[\theta-\theta_i]$ is a pid and we can apply Rouchaleau's algorithm [11] to compute minimal decompositions over $R[\theta-\theta_i]$. The procedure is as follows.

Let $(F,G,H)_n$ be a minimal decomposition of $W \in R(\theta_1,\theta_2)^{k\times m}$ over $R(\theta-\theta_i)$, computed from the matrix J in the Silverman procedure. Since $R[\theta-\theta_i]$ is a pid, it follows that there exists a $n\times n$ invertible matrix T over $R(\theta-\theta_i)$ such that $F = T^{-1}\bar{F}T$, $G = T^{-1}\bar{G}$, $H = \bar{H}T$ is a minimal decomposition of W over $R[\theta-\theta_i]$. To compute T, let N be the $n\times\sigma m$ submatrix of the Hankel matrix $\Gamma_{\sigma,\sigma}$ containing the same rows as J. We then find a basis for the columns of N over the pid $R[\theta-\theta_i]$:

Let π_1 be the greatest common divisor of the elements in the first row of N. There is a linear combination γ_1 over $R[\theta-\theta_i]$ of the columns of N having π_1 as first element, and for each column γ of N, there exists an $\alpha \in R[\theta-\theta_i]$ such that the first element of $\gamma-\alpha\gamma_1$ is zero. Doing this for each column of N, we obtain a matrix N_1 such that γ_1 and the columns of N_1 generate the columns of N over $R[\theta-\theta_i]$. Applying this procedure to N_1, and so on, we obtain a matrix $(\gamma_1,\ldots,\gamma_n)$ such that $\gamma_1,\ldots,\gamma_n$ generate the columns of N over $R[\theta-\theta_i]$ and $T = J^{-1}(\gamma_1,\ldots,\gamma_n)$. An example of this construction is given in the next section.

In the general case, the question of the uniqueness of minimal decompositions over $R[\theta-\theta_i]$ appears to be difficult to answer; we leave this as an open problem. However, we do have the following result.

<u>Theorem 7</u>: If $q = 2$ and θ_1 and θ_2 are algebraically independent over R, then given any two minimal decompositions $(F,G,H)_n$ and $(\hat{F},\hat{G},\hat{H})_n$ over $R[\theta-\theta_i]$ of $W \in R(\theta_1,\theta_2)^{k\times m}$ there exists an $n\times n$ invertible matrix A over $R(\theta-\theta_i)$ such that $\hat{F} = AFA^{-1}$, $\hat{G} = AG$, $\hat{H} = HA^{-1}$.

<u>Proof</u>: Let $(F,G,H)_n$ be a minimal decomposition over $R[\theta-\theta_i]$. We claim that $(F,G,H)_n$ is also a minimal decomposition over $R(\theta-\theta_i)$, for suppose it is not. Then there exists $(\bar{F},\bar{G},\bar{H})_m$, $m<n$, which is a minimal decomposition over $R(\theta-\theta_i)$. By Rouchaleau's results [11], from $(\bar{F},\bar{G},\bar{H})_m$ we can construct a decomposition over $R[\theta-\theta_i]$ of size m, a contradiction. Now given two minimal decompositions

over $R[\theta-\theta_i]$, since they are also minimal over $R(\theta-\theta_i)$, by Theorem 5 we have the desired result.

7. An Example

Consider a delay-differential system whose input/output operator f is given by $f:U^2 \to V^2 : u \mapsto W*u$ where

$$W = \frac{1}{p^2+pd} \begin{pmatrix} 2d^2p & -6 \\ -2d^3p & -2p+4d \end{pmatrix} \in R(p,d)^{2\times 2}, \quad p = \delta_o^{(1)}, \quad d = \delta_1$$

It is seen that W satisfies the hypothesis of the corollary to Theorem 6 with $\theta_i = p$, and thus, it has a minimal decomposition over $R[d]$ which we now compute. Since the degree of p^2+pd, viewed as an element of $R[d][p]$, is two, we need to consider the Handel matrix $\Gamma_{2,2}$. Expanding the elements of W, we obtain:

$$\Gamma_{2,2} = \begin{pmatrix} 2d^2 & 0 & -2d^3 & -6 \\ -2d^3 & -2 & 2d^4 & 6d \\ -2d^3 & -6 & 2d^4 & 6d \\ 2d^4 & 6d & -2d^5 & -6d^2 \end{pmatrix}$$

The rank of $\Gamma_{2,2}$ (as a matrix over $R(d)$) is two. We then pick

$$J = \begin{pmatrix} 2d^2 & 0 \\ -2d^3 & -2 \end{pmatrix} \qquad \text{with } J^{-1} = \begin{pmatrix} \frac{1}{2d^2} & 0 \\ -d/2 & -1/2 \end{pmatrix}$$

We then have that $K = J$, $L = J$, and

$$M = \begin{pmatrix} -2d^3 & -6 \\ 2d^4 & 6d \end{pmatrix}$$

which gives the following decomposition over $R(d)$

$$\overline{F} = \begin{pmatrix} -d & \frac{-3}{d^2} \\ 0 & 0 \end{pmatrix}, \quad \overline{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{H} = \begin{pmatrix} 2d^2 & 0 \\ -2d^3 & -2 \end{pmatrix}$$

Now

$$N = \begin{pmatrix} 2d^2 & 0 & -2d^3 & -6 \\ -2d^3 & -2 & 2d^4 & 6d \end{pmatrix}$$

and via the procedure given above, we find that $\gamma_1 = \begin{pmatrix} -1 \\ d \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

generate the columns of N over R[d].

Hence,

$$T = J^{-1}\ (\gamma_1,\gamma_2) = \begin{pmatrix} \frac{-1}{2d^2} & 0 \\ 0 & 1 \end{pmatrix}$$

Then,

$$F = T^{-1}\overline{F}T = \begin{pmatrix} -d & 6 \\ 0 & 0 \end{pmatrix}, \qquad G = T^{-1}\overline{G} = \begin{pmatrix} -2d^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $H = \overline{H}T = \begin{pmatrix} -1 & 0 \\ d & -2 \end{pmatrix}$

Since $R(p,d) \subset V$ by Proposition 4, the decomposition $(F,G,H)_n$ over R[d] yields a realization of minimal size of the input/output operator f. In component form the realization is given by

$$\frac{dx_1(t)}{dt} = -x_1(t-1) + 6x_2 - 2u_1(t-2)$$

$$\frac{dx_2(t)}{dt} = u_2(t)$$

$$y_1(t) = -x_1(t)$$

$$y_2(t) = x_1(t-1) - 2x_2(t)$$

References

1. L. Schwartz, Theorie des Distributions, Hermann, Paris, 1966.

2. H. Blomberg, J. Sinervo, A. Halme, and R. Ylinen, "On Algebraic Methods in Systems Theory," ACTA Poly. Scandinavica, Series #19, Helsinki, 1969.

3. J. Mikusiński, Operational Calculus, Pergamon Press, New York, 1959.

4. O. Zariski and P. Samuel, Commutative Algebra, Volume 1, Van Nostrand, Princeton, 1958.

5. R. Bellman and K. Cooke, Differential-Difference Equations, Academic Press, 1963.

6. M. Oğuztöreli, Time-Lag Control Systems, Academic Press, 1966.

7. R. Kalman, P. Falb, and M. Arbib, Topics in Mathematical System Theory, McGraw-Hill, 1969.

8. E. Kamen, "A Distributional-Module Theoretic Representation of Linear Continuous-Time Systems," Rept. SEL-71-044 (TR.No. 6560-24), Stanford Electronics Labs, Stanford, Calif., 1971.

9. B. Ho, "On Effective Construction of Realizations from Input-Output Description," Ph.D. Dissertation, Stanford Univ., 1966.

10. R. Newcomb, "On the Realization of Multivariable Transfer Functions," Research Report EERL 58, Cornell Univ., 1966.

11. Y. Rouchaleau, "Linear, Discrete Time, Finite Dimensional, Dynamical Systems Over Some Classes of Commutative Rings," Ph.D. Dissertation, Stanford Univ., 1972.

12. Y. Rouchaleau, B. Wyman, and R. Kalman, "Algebraic Structure of Linear Dynamical Systems. III. Realization Theory Over a Commutative Ring," Proc. Natl. Aca. Science, Vol. 69, No. 11, pp. 3404-3406, Nov. 1972.

13. P. Cahen and J. Chabert, "Eléments quasi-entiers et extensions de Fatou," Queen's Math. Preprint No. 1972-22, Queen's University, Ontario, Canada, 1972.

14. Y. Rouchaleau and B. Wyman, "Linear Dynamical Systems over Integral Domains," to appear.

A NEW ALGEBRAIC APPROACH TO LINEAR TIME-VARYING SYSTEMS*

Edward W. Kamen

School of Electrical Engineering

Georgia Institute of Technology

Atlanta, Georgia 30332

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ABSTRACT

A theory of linear time-varying discrete-time systems is constructed in terms of a variable time reference which yields a new type of global-in-time representation. In this approach the time-variance of systems is incorporated into an algebraic framework consisting of modules defined over noncommutative rings. In particular, input/output behavior is specified by a homomorphism between modules over a noncommutative ring of formal power series, yielding an operational calculus for computing system responses. Dynamical behavior is given in terms of a module structure defined over a skew polynomial ring. This framework is utilized to obtain general results on reachability and controllability, and is then applied to the problem of realizing time-varying discrete-time systems.

1. Introduction

The existing state space theory of linear time-varying systems is based primarily on the pointwise-in-time formulation; that is, for each instant of time the system dynamics are given in terms of linear transformations between linear spaces over a field of scalars. Although many deep results have been obtained by using the linear space representation (see [1]), in the time-varying case it is not well suited to the study of global-time behavior since the pointwise-in-time framework does not clearly characterize the time variance of a given class of systems. However, the time variance can be taken into account in a direct fashion by viewing the coefficient matrices of the state equations as matrices over some ring of time functions. This viewpoint is utilized in [2] to study the structure of a class of linear time-varying continuous-time systems.

In this paper, we present a theory of linear time-varying discrete-time systems which yields global-time results by incorporating the time-variance of systems into the algebraic structure. In contrast to previous approaches, the theory is developed in terms of a type of global-in-time representation specified by a variable time reference. As will be demonstrated, the global-in-time description possesses previously unexplored algebraic properties which play an important role in system dynamics. These properties are incorporated into an algebraic framework via a module structure defined over a noncommutative ring of polynomials.

Noncommutative polynomials in the form of differential polynomials with time-varying coefficients have been utilized in the study of the input/output representation of time-varying continuous-time systems (see Chapter 6 of the book written by Zadeh and Desoer [3]). Recently, Salovaara and Blomberg [4] applied the differential polynomial formulation to the study of time-varying differential systems with stochastic processes as inputs and outputs.

However, these approaches are not based to any great extent on the algebraic properties of noncommutative polynomial rings. The first (and thus far the only) deep application of this algebraic structure in engineering appeared in the work of Newcomb [5]. Using Ore's theory [6] of noncommutative polynomials, Newcomb developed an operational method for a local time-variable synthesis of networks.

Until the present work, the theory of noncommutative polynomial rings had not been applied to the state space description of time-varying systems. Many factors suggest that this should be possible, the most obvious of which is Kalman's module framework over the usual commutative ring of polynomials applied to linear time-invariant discrete-time systems [7]. The construction of a module structure in the time-varying case is mainly a problem of generating the correct formulation which is done in Section 2. The crucial step which yields a module structure over a noncommutative ring of polynomials is accomplished by extending the notion of a semi-linear transformation [8]. This is given in Section 3. In Sections 4 and 5, the module framework is utilized to obtain results on control and realization.

2. System Definition

Let Z denote the ring of integers and let R be a commutative ring with 1. In the sequel, all R-modules are right R-modules and all module homomorphisms are written on the right.

Definition 1: A linear discrete-time dynamical system Σ over R is a triple X,U,Y of R-modules, together with three collections of R-modules homomorphisms

$$\{F_k : X \to X,\ k \in Z\}$$

$$\{G_k : U \to X,\ k \in Z\}$$

$$\{H_k : X \to Y,\ k \in Z\}$$

In the usual manner, a system $\Sigma = (X,U,Y;F_k,G_k,H_k)$ defines the following dynamical equations

$$x_{k+1} = x_k F_k + u_k G_k \qquad k \in Z \tag{1.1}$$

$$y_k = x_k H_k \tag{1.2}$$

where the state x_k belongs to the state module X, the input u_k belongs to the input module U, and the output y_k belongs to the output module Y.

In contrast to the pointwise-in-time representation given by (1.1-2), the objective here is to formulate a type of global-in-time description of time-varying discrete-time systems. To do this, we first need the following constructions.

Given a commutative ring R with 1, let R^Z denote the set of all functions $\alpha: Z \to R$. With pointwise addition and multiplication, R^Z is a commutative ring with 1. Let σ denote the right-shift operator on R^Z defined by $(\alpha\sigma)(k) = \alpha(k-1)$. Since σ is a ring automorphism, the ring R^Z with σ is a difference ring.

Given an R-module W, let W^Z denote the (right) R^Z-module of all functions $\underset{\sim}{w}:Z \to W$ with pointwise addition and with multiplication $W^Z \times R^Z \to W^Z : (\underset{\sim}{w},\alpha) \mapsto \underset{\sim}{w}\alpha$ defined by $(\underset{\sim}{w}\alpha)(k) = \underset{\sim}{w}(k)\alpha(k)$. Note that if W is a finitely-generated R-module with generators $w_1, w_2, \ldots, w_n$, and if we define the constant functions

$$\bar{w}_i : Z \to W : k \mapsto w_i, \quad i = 1,2,\ldots,n$$

then $W^Z = \langle \bar{w}_1,\ldots,\bar{w}_n \rangle_{R^Z}$.

Let $A \neq \{0\}$ be a fixed subring of R^Z with $(A)\sigma = A$; that is, σ is a ring automorphism on A. Note that A contains all the constant functions $\bar{a}:Z \to R : k \mapsto a,\ a \in R$.

Then given W , let $\bar{W}_C = \{\bar{w}:Z \to W : k \mapsto w,\ w \in W\}$ and define

$$\bar{W}_A = \left\{ \sum_{r=o}^{m} \bar{w}_r \alpha_r : \bar{w}_r \in \bar{W}_C, \alpha_r \in A,\ Z \ni m \geqq o \right\} \tag{2}$$

Clearly, $\bar{W}_A$ is an A-submodule of W^Z viewed as an A-module. Furthermore, since A contains all the constant functions $Z \to R$, if W is finitely generated, for any set $\{w_1,\ldots,w_n\}$ of generators of W, we have that $\bar{W}_A = \langle \bar{w}_1,\ldots,\bar{w}_n \rangle_A$.

Definition 2: Given R-modules V and W, an R^Z-module homomorphism $E:V^Z \to W^Z$ is said to be closed with respect to A if $(\bar{V}_A)E \subset \bar{W}_A$.

In the finitely-generated case, we have the following characterization of homomorphisms closed with respect to A.

Proposition 1: If V and W are finitely-generated R-modules, $E:V^Z \to W^Z$ is closed with respect to A if and only if with respect to any set of generators of V^Z and W^Z consisting of constant functions, the matrix representation of E is over A.

Proof: Clear.

Now given a system $\Sigma = (X,U,Y;F_k,G_k,H_k)$, define the operators

(3.1) $$F:X^Z \mapsto X^Z : \underset{\sim}{x} \mapsto \underset{\sim}{x}F : k \rightsquigarrow \underset{\sim}{x}(k)F_k$$

(3.2) $$G:U^Z \mapsto X^Z : \underset{\sim}{u} \mapsto \underset{\sim}{u}G : k \rightsquigarrow \underset{\sim}{u}(k)G_k$$

(3.3) $$H:X^Z \mapsto Y^Z : \underset{\sim}{x} \mapsto \underset{\sim}{x}H : k \rightsquigarrow \underset{\sim}{x}(k)H_k$$

It is obvious that F,G, and H are R^Z-module homomorphisms. We can therefore give the following

Definition 3: Let A be a subring of R^Z with $(A)\sigma = A$. A system Σ is said to be a closed A-system if the R^Z-module homomorphisms defined by (3.1-3) are closed with respect to A.

By Proposition 1, if X,U, and Y are finitely-generated R-modules, then Σ is a closed A-system if and only if with respect to any set of generators of X^Z, U^Z, and Y^Z consisting of constant functions, the matrix representations of the operators F,G, and H are over A. Note that if X, U, and Y are finite and A is the subring of R^Z consisting of all constant functions $Z \rightarrow R$, then a closed A-system is a time-invariant system.

Via the concept of a closed A-system, we can restrict attention to particular classes of time-varying systems possessing an enriched algebraic structure resulting from various ring-theoretic properties enjoyed by A. For example, if R is a field and A is the ring R[k] of polynomials in time with the usual operations, then A is a principal ideal domain. The richness of this algebraic structure from a computational standpoint is clear. The special theoretical aspects of this case (and others) will be developed in a separate paper, as our objective here is to present the theory of closed A-systems in full generality.

The following property of closed A-systems will be needed shortly.

Proposition 2: Let $\Sigma = (X,Y,U;F_k,G_k,H_k)$ be a closed A-system. Then for each fixed $i \in Z$, the following operators are A-module homomorphisms

(4.1) $$\bar{X}_A \to \bar{X}_A : \underset{\sim}{x} \mapsto \underset{\sim}{x}(\cdot)F_{(\cdot)+i} : j \rightsquigarrow \underset{\sim}{x}(j)F_{j+i}$$

(4.2) $$\bar{U}_A \to \bar{X}_A : \underset{\sim}{u} \mapsto \underset{\sim}{u}(\cdot)G_{(\cdot)+i} : j \rightsquigarrow \underset{\sim}{u}(j)G_{j+i}$$

(4.3) $$\bar{X}_A \to \bar{Y}_A : \underset{\sim}{x} \mapsto \underset{\sim}{x}(\cdot)H_{(\cdot)+i} : j \rightsquigarrow \underset{\sim}{x}(j)H_{j+i}$$

Proof: Since σ is a ring automorphism on A, the operator $\sigma_X : \bar{X}_A \mapsto \bar{X}_A : \underset{\sim}{x} \to \underset{\sim}{x}\sigma_X : k \rightsquigarrow \underset{\sim}{x}(k-1)$ is an automorphism with respect to the group structure on $\bar{X}_A$. By definition of F (see (3.1)), for any $\underset{\sim}{x} \in \bar{X}_A$ we have that $\underset{\sim}{x}(j)F_{j+i} = (\underset{\sim}{x}\sigma_X^{i}F\sigma_X^{-i})(j)$; and since $(\bar{X}_A)F \subset \bar{X}_A$, the function $Z \to X : j \mapsto \underset{\sim}{x}(j)F_{j+i}$ is indeed an element of $\bar{X}_A$ for every $i \in Z$. Hence the operator given by (4.1) is properly defined and it is easily verified that it is an A-module homomorphism. The proof for (4.2-3) is similar and is therefore omitted.

We now develop a framework for the study of closed A-systems. We begin with the following construction.

Given an R-module W, construct $\bar{W}_A$ as defined by (2). Let $S(\bar{W}_A)$ denote the set of all functions $\underset{\sim}{w} : Z \times Z \to W$ such that

(5) $$\begin{aligned} &\underset{\sim}{w}(j+i,j) \triangleq \underset{\sim}{w}_i(j), \underset{\sim}{w}_i \in \bar{W}_A, \quad i = q,q+1,\ldots \\ &\underset{\sim}{w}(j+i,j) = 0, \ i < q \end{aligned}$$

where q is some integer which may depend on $\underset{\sim}{w}$.

The interpretation of this setup is that any element $\underset{\sim}{w} \in S(\bar{W}_A)$ can be viewed as a collection of functions $\{\underset{\sim}{w}(\cdot,j):Z \to W,\ j \in Z\}$ with j acting as a time reference. Note that each $\underset{\sim}{w}(\cdot,j)$ has support bounded on the left.

<u>Proposition 3</u>: $S(\bar{W}_A)$ is a (right) A-module with pointwise addition and with multiplication $S(\bar{W}_A) \times A \to S(\bar{W}_A):(\underset{\sim}{w},\alpha) \mapsto \underset{\sim}{w}\alpha$ defined by $(\underset{\sim}{w}\alpha)(j+i,j) = \underset{\sim}{w}(j+i,j)\alpha(j)$.

<u>Proof</u>: Clear

Now let $\Sigma = (X,U,Y;F_k,G_k H_k)$ be a closed A-system and construct $S(\bar{X}_A)$, $S(\bar{U}_A)$, and $S(\bar{Y}_A)$ as defined by (5). In addition to the dynamical equations (1.1-2), we also have the following equations

(6.1) $$\underset{\sim}{x}(j+i+1,j) = \underset{\sim}{x}(j+i,j)F_{j+i} + \underset{\sim}{u}(j+i,j)G_{j+i}$$

(6.2) $$\underset{\sim}{y}(j+i,j) = \underset{\sim}{x}(j+i,j)H_{j+i}$$

where $\underset{\sim}{x} \in S(\bar{X}_A)$, $\underset{\sim}{u} \in S(\bar{U}_A)$, $\underset{\sim}{y} \in S(\bar{Y}_A)$, and for each fixed $j \in Z$, $\underset{\sim}{x}(j+i,j) \in X$ (respectively, $\underset{\sim}{u}(j+i,j) \in U, \underset{\sim}{y}(j+i,j) \in Y$) is the state (resp. input, output) at time j+i.

In (6.1-2), j acts as a variable time reference. More precisely, for each fixed $j \in Z$, the equations (6.1-2) describe the dynamical behavior of the system Σ in response to the input $\underset{\sim}{u}(\cdot,j):Z \to U$.

The representation given by (6.1-2) may appear to be unnecessarily complicated. However, as we now proceed to show, by starting from this framework we can construct a new algebraic theory of time-varying systems.

First we solve the equations (6.1-2). Consider the collection of R-module homomorphisms $\{L_{j+i,j}:X \to X,\ j \in Z,\ i \geq -1\}$ defined by

$$L_{j+i,j} = F_j F_{j+1} \cdots F_{j+i},\ \text{all } j,\ i \geq 0$$

$$L_{j-1,j} = \text{identity operator on } X, \text{ all } j$$

Then by iteration, the solution $\underset{\sim}{x}(j+i,j)$ of (6.1) at time j+i starting from initial state $\underset{\sim}{x}(j+q,j)$, $q < i$, is

(7) $$\underset{\sim}{x}(j+i,j) = \underset{\sim}{x}(j+q,j)L_{j+i-1,j+q} + \sum_{r=q}^{i-1} \underset{\sim}{u}(j+r,j)G_{j+r}L_{j+i-1,j+r+1}$$

If $\underset{\sim}{x}(j+q,j) = 0$, all j, then

(8) $$\underset{\sim}{y}(j+i,j) = \underset{\sim}{x}(j+i,j)H_{j+i} = \sum_{r=q}^{i-1} \underset{\sim}{u}(j+r,j)G_{j+r}L_{j+i-1,j+r+1}H_{j+i}$$

In (7-8), if we write $\underset{\sim}{x}(j+i,j) = \underset{\sim}{x}_i(j)$, $\underset{\sim}{u}(j+i,j) = \underset{\sim}{u}_i(j)$, and $\underset{\sim}{y}(j+i,j) = \underset{\sim}{y}_i(j)$, then since $\underset{\sim}{u}_i \in \bar{U}_A$ and $\sum$ is a closed A-system, it follows from Proposition 2 that $\underset{\sim}{x}_i \in \bar{X}_A$ and $\underset{\sim}{y}_i \in \bar{Y}_A$ for every $i \in Z$.

Now define the operator $f_\Sigma : S(\bar{U}_A) \to S(\bar{Y}_A) : \underset{\sim}{u} \mapsto \underset{\sim}{u}f_\Sigma$ where

(9) $$\left(\underset{\sim}{u}f_\Sigma\right)(j+i,j) = \sum_{r=-\infty}^{i-1} \underset{\sim}{u}(j+r,j)G_{j+r}L_{j+i-1,j+r+1}H_{j+i}$$

By definition of $S(\bar{U}_A)$, given $\underset{\sim}{u} \in S(\bar{U}_A)$ there exists an integer q such that $\underset{\sim}{u}(j+i,j) = 0$, $i < q$. Hence the sum in (9) is finite and $(\underset{\sim}{u}f_\Sigma)(j+i,j) = 0$, $i < q+1$.

A comparison between (8) and (9) reveals that for any $\underset{\sim}{u} \in S(\bar{U}_A)$, $\underset{\sim}{u}f_\Sigma$ is the collection of responses of the system resulting from the collection of inputs $\left\{\underset{\sim}{u}(\cdot,j) : Z \to U,\ j \in Z\right\}$ with zero initial state prior to application of each input. In particular, for each fixed $j \in Z$, $(\underset{\sim}{u}f_\Sigma)(\cdot,j) : Z \to Y$ is the response to the input $\underset{\sim}{u}(\cdot,j) : Z \to U$ with zero initial state. Hence f_Σ is a (collective) input/output operator of the system $\sum$. Note that since A contains all the constant functions from Z into R, $S(\bar{U}_A)$ contains all functions from Z into U with support bounded on the left. Thus f_Σ completely characterizes the input/output behavior of the system $\sum$.

With respect to the A-module structure on $S(\bar{U}_A)$ and $S(\bar{Y}_A)$ as given in Proposition 3, it is easily verified that f_Σ is an A-module homomorphism. The operator f_Σ also commutes with shifts in the time reference defined in the following manner.

Given an R-module W, construct $S(\bar{W}_A)$ and define the operator $\rho_W: S(\bar{W}_A) \to S(\bar{W}_A): \underset{\sim}{w} \mapsto \underset{\sim}{w}\rho_W$ where

$$(\underset{\sim}{w}\rho_W)(j+i,j) = \underset{\sim}{w}(j+i,j-1) \quad . \tag{10}$$

The operator ρ_W has the following interpretation. Given $\underset{\sim}{w} \in S(\bar{W}_A)$, viewed as a collection of functions $\{\underset{\sim}{w}(\cdot,j): Z \to W,\ j \in Z\}$, by definition of ρ_W, for each fixed $j \in Z$, the function $(\underset{\sim}{w}\rho_W)(\cdot,j): Z \to W$ is equal to the function $\underset{\sim}{w}(\cdot,j-1): Z \to W$. Hence ρ_W produces a shift in the time reference j.

Now define $\rho_U: S(\bar{U}_A) \to S(\bar{U}_A)$ and $\rho_Y: S(\bar{Y}_A) \to S(\bar{Y}_A)$ as given by (10), then we have

<u>Proposition 4</u>: $(\underset{\sim}{u}\rho_U)f_\Sigma = (\underset{\sim}{u}f_\Sigma)\rho_Y$, all $\underset{\sim}{u} \in S(\bar{U}_A)$

Proof: For each $j \in Z$, since $(\underset{\sim}{u}f_\Sigma)(\cdot,j)$ is the zero state response to $\underset{\sim}{u}(\cdot,j)$, replacing j by j-1 gives the desired result.

The commutativity of f_Σ with time-reference shifts is indeed a trivial property, and yet this result is very significant as revealed in the next section.

3. Module Structure of Time-Varying Systems

Given the input/output operator f_Σ defined by (9), in the first part of this section it is shown that the properties of f_Σ can be incorporated into a module structure defined over a noncommutative ring of power series. This is accomplished by means of the following constructions.

Given a commutative ring R with 1, as before let A be a subring of R^Z with $(A)\sigma = A$. Letting z be an indeterminate, define

$$A((z^{-1})) = \left\{ \sum_{r=-N}^{\infty} z^{-r}\alpha_r : \alpha_r \in A,\ N \in Z \right\}$$

With the usual addition and with multiplication defined by

$$\begin{aligned} z^r z^s &= z^{r+s}\ , \quad r,s \in Z \\ \alpha z^r &= z^r(\alpha\sigma^r),\ \alpha \in A,\ r \in Z \end{aligned} \tag{11}$$

$A((z^{-1}))$ is a noncommutative ring, called the skew ring of formal Laurent series over A with coefficients written on the right.

Given an R-module W, construct $\bar{W}_A$ as defined by (2) and let $\bar{W}_A((z^{-1}))$ denote the (right) $A((z^{-1}))$-module given by

$$\bar{W}_A((z^{-1})) = \left\{ \sum_{r=-N}^{\infty} z^{-r} \underset{\sim}{w}_r : \underset{\sim}{w}_r \in \bar{W}_A,\ N \in Z \right\}$$

Multiplication in $\bar{W}_A((z^{-1}))$ is defined by

$$(12) \qquad \begin{aligned} z^r z^s &= z^{r+s} \\ \underset{\sim}{w}_r z &= z(\underset{\sim}{w}_r \sigma_W),\ \underset{\sim}{w}_r \in \bar{W}_A \end{aligned}$$

where $\sigma_W : \bar{W}_A \to \bar{W}_A : \underset{\sim}{w} \mapsto (\underset{\sim}{w}\sigma_W) : k \rightsquigarrow \underset{\sim}{w}(k-1)$

Now construct $S(\bar{W}_A)$ and define $\lambda : S(\bar{W}_A) \to \bar{W}_A((z^{-1})) : \underset{\sim}{w} \to \sum_{i=-\infty}^{\infty} z^{-i} \underset{\sim}{w}_i$

where $\underset{\sim}{w}_i(j) = \underset{\sim}{w}(j+i,j)$, all $i,j \in Z$.

<u>Proposition 5</u>: The operator λ is an A-module isomorphism. Further, for every $\underset{\sim}{w} \in S(\bar{W}_A)$, $(\underset{\sim}{w}\rho_W)\lambda = (\underset{\sim}{w}\lambda)z$ where ρ_W is the time-reference shift defined by (10).

<u>Proof</u>: It is clear that λ is an A-module isomorphism with $\bar{W}_A((z^{-1}))$ viewed as an A-module. Now let $\underset{\sim}{w} \in S(\bar{W}_A)$ and write $\underset{\sim}{w}(j+i,j) = \underset{\sim}{w}_i(j)$. Then

$$(\underset{\sim}{w}\rho_W)(j+i,j) = \underset{\sim}{w}(j+i,j-1) = \underset{\sim}{w}\Big((j-1) + (i+1),\ j-1\Big)$$

$$\Rightarrow (\underset{\sim}{w}\rho_W)(j+i,j) = \underset{\sim}{w}_{i+1}(j-1) = (\underset{\sim}{w}_{i+1}\sigma_W)(j)$$

Thus

$$(\underset{\sim}{w}\rho_W)\lambda = \sum_{i=-\infty}^{\infty} z^{-i}\left(\underset{\sim}{w}_{i+1}\sigma_W\right)$$

Replacing i by i-1 yields

$$(\underset{\sim}{w}\rho_W)\lambda = \sum_{i=-\infty}^{\infty} z^{-i}z(\underset{\sim}{w}_i\sigma_W) = \sum_{i=-\infty}^{\infty} (z^{-i}\underset{\sim}{w}_i)z \quad \text{by (12)}$$

Hence $(\underset{\sim}{w}\rho_W)\lambda = (\underset{\sim}{w}\lambda)z$.

As a consequence of Proposition 5, we can represent elements in $S(\bar{W}_A)$ by Laurent series in z^{-1}. This construction is similar to that given by Kalman [7] and Wyman [9] in the representation of scalar sequences in terms of an indeterminate. The new aspect of the above framework is that it applies to time-varying systems, and in fact it yields a time-varying operational calculus, as we shall see later.

Now let $\sum$ be a closed A-system with input/output operator $f_\Sigma : S(\bar{U}_A) \to S(\bar{Y}_A)$. Since $\sum$ and A are fixed, we shall write $\mathcal{U} = \bar{U}_A$ and $\mathcal{Y} = \bar{Y}_A$, and omit the subscript $\sum$ on f, so that we have $f : S(\mathcal{U}) \to S(\mathcal{Y})$. Representing the elements of $S(\mathcal{U})$ and $S(\mathcal{Y})$ by the corresponding elements in $\mathcal{U}((z^{-1}))$ and $\mathcal{Y}((z^{-1}))$, respectively, we can view f as an operator from $\mathcal{U}((z^{-1}))$ into $\mathcal{Y}((z^{-1}))$. In terms of this framework, we have the following results on the properties of f.

<u>Lemma</u>: For every $\underset{\sim}{u} = \sum_{r=-\infty}^{\infty} z^{-r}\underset{\sim}{u}_r \in \mathcal{U}((z^{-1}))$

$$\underset{\sim}{u}f = \sum_{r=-\infty}^{\infty} \left((\underset{\sim}{u}_r\sigma_U^r)f\right)z^{-r}$$

where $\sigma_U^r : \mathcal{U} \to \mathcal{U} : \underset{\sim}{u} \mapsto (\underset{\sim}{u}\sigma_U) : k \rightsquigarrow \underset{\sim}{u}(k-r)$, $r \in Z$

Proof: Given $\underset{\sim}{u} = \sum_{r=-\infty}^{\infty} z^{-r} \underset{\sim}{u}_r$, for each $r \in Z$, write

$$(\underset{\sim}{u}_r \sigma_U^r) f = \sum_{i=-\infty}^{\infty} z^{-i} \underset{\sim}{y}_{r,i} \text{ , } \underset{\sim}{y}_{r,i} \in \mathcal{Y} \text{ .}$$

By the definition of f,

$$(13) \qquad \underset{\sim}{y}_{r,i}(j) = \begin{cases} (\underset{\sim}{u}_r \sigma_U^r)(j) G_j L_{j+i-1,j+1} H_{j+i} \text{ , } & i > 0 \\ 0 \text{ , } & i \le 0 \end{cases}$$

By definition of multiplication in the $A((z^{-1}))$-module $\mathcal{Y}((z^{-1}))$, we have

$$((\underset{\sim}{u}_r \sigma_U^r) f) z^{-r} = \sum_{i=-\infty}^{\infty} \left(z^{-i} \underset{\sim}{y}_{r,i}\right) z^{-r}$$

$$= \sum_{i=-\infty}^{\infty} z^{-i-r} \left(\underset{\sim}{y}_{r,i} \sigma_Y^{-r}\right)$$

$$= \sum_{i=-\infty}^{\infty} z^{-i} \left(\underset{\sim}{y}_{r,i-r} \sigma_Y^{-r}\right)$$

where $\sigma_Y^{-r} : \mathcal{Y} \to \mathcal{Y} : \underset{\sim}{y} \to \underset{\sim}{y} \sigma_Y^{-r} : k \rightsquigarrow \underset{\sim}{y}(k+r)$

Then from (13),

$$\left(\underset{\sim}{y}_{r,i-r} \sigma_Y^{-r}\right)(j) = \begin{cases} \underset{\sim}{u}_r(j) G_{j+r} L_{j+i-1,j+r+1} H_{j+i} \text{ , } & i > r \\ 0 \text{ , } & i \le r \end{cases}$$

Hence using the definition of f, we have

$$\sum_r \left(\left(\underset{\sim}{u}_r \sigma_U^r\right) f\right) z^{-r} = \sum_i z^{-i} \left(\sum_r \underset{\sim}{y}_{r,i-r} \sigma_Y^{-r} \right) = \underset{\sim}{u} f$$

The above lemma shows that the operator f is completely determined by its action on $\mathcal{U}$. Using this result, we can prove the following:

Theorem 1: f is an $A((z^{-1}))$-module homomorphism.

Proof: The only point which is nontrivial is the proof that f commutes with elements in $A((z^{-1}))$. First we show that

(14) $$(\underset{\sim}{u}z^{r})f = (\underset{\sim}{u}f)z^{r} \quad \text{for all } r \in Z$$

Let $\underset{\sim}{u} \in \mathcal{U}((z^{-1}))$, then $(\underset{\sim}{u}f)z^{-1} = (\underset{\sim}{u}z^{-1}z)fz^{-1}$, and by Propositions 4 and 5, $(\underset{\sim}{u}z^{-1}z)fz^{-1} = (\underset{\sim}{u}z^{-1})fzz^{-1}$. Thus $(\underset{\sim}{u}f)z^{-1} = (\underset{\sim}{u}z^{-1})f$, and by repeated application of z and z^{-1}, we have (14). Now let $\underset{\sim}{u} = \sum_{r} z^{-r}\underset{\sim}{u}_r \in \mathcal{U}((z^{-1}))$, then

$$\underset{\sim}{u}f = \sum_{r}\left((\underset{\sim}{u}_r\sigma_U^{r})f\right)z^{-r} \quad \text{by the lemma}$$

$$\underset{\sim}{u}f = \sum_{r}\left((\underset{\sim}{u}_r\sigma_U^{r})z^{-r}\right)f \quad \text{by (14)}$$

(15) $$\underset{\sim}{u}f = \sum_{r}\left(z^{-r}\underset{\sim}{u}_r\right)f, \text{ by definition of multiplication in } \mathcal{U}((z^{-1}))$$

Let $\gamma = \sum_{i=-\infty}^{\infty} z^{-i}\gamma_i \in A((z^{-1}))$, then

$$(\underset{\sim}{u}\gamma)f = \left(\sum_{i}\left(\sum_{r} z^{-r}\underset{\sim}{u}_r\right)z^{-i}\gamma_i\right) f$$

$$(\underset{\sim}{u}\gamma)f = \left(\sum_{i}\sum_{r} z^{-r-i}\left(\underset{\sim}{u}_r\sigma_U^{-i}\right)\gamma_i\right)f$$

Applying (15) twice and using the A-linearity of f, we obtain

$$(\underset{\sim}{u}\gamma)f = \sum_{i}\sum_{r}\left(z^{-r-i}\left(\underset{\sim}{u}_r\sigma_U^{-i}\right)\right)f\gamma_i$$

Then

$$(\underset{\sim}{u}\gamma)f = \sum_{i}\left(\sum_{r}\left(\underset{\sim}{u}_r\sigma_U^{r}\right)fz^{-r}\right) z^{-i}\gamma_i \quad \text{by (14)}$$

$$(\underset{\sim}{u}\gamma)f = \sum_{i} (\underset{\sim}{u}f)z^{-i}\gamma_i \quad \text{by the lemma}$$

$$(\underset{\sim}{u}\gamma)f = (\underset{\sim}{u}f)\gamma$$

When the input module U and the output module Y are finitely-generated R-modules, it follows from Theorem 1 that there exists an operational calculus for computing system responses. In particular, suppose the set of elements $\{c_1,\ldots,c_m\}$ generates U and the set $\{d_1,\ldots,d_p\}$ generates Y. Then as an A-module, $\mathcal{U} = \bar{U}_A$ and $\mathcal{Y} = \bar{Y}_A$ are generated by the constant functions $\{\bar{c}_1,\ldots,\bar{c}_m\}$ and $\{\bar{d}_1,\ldots,\bar{d}_p\}$, respectively. It is clear that these sets also generate $\mathcal{U}((z^{-1}))$ and $\mathcal{Y}((z^{-1}))$ as $A((z^{-1}))$-modules. Then since f is an $A((z^{-1}))$-module homomorphism, with respect to these sets of generators f can be represented by a $p \times m$ matrix Θ over $A((z^{-1}))$. That is, given $\underset{\sim}{u} = \sum_{r=1}^{m} \bar{c}_r u_r$ with $\underset{\sim}{u}f = \sum_{r=1}^{p} \bar{d}_r y_r$, then writing $\underset{\sim}{u} = (u_1,\ldots,u_m)^T$, $\underset{\sim}{y} = (y_1,\ldots,y_p)^T$, we have

$$\underset{\sim}{y} = \Theta \underset{\sim}{u} \tag{16}$$

The multiplication in (16) is the usual multiplication of a matrix by a column vector with componentwise operation carried out in $A((z^{-1}))$. Note that in contrast to the standard operational setup in the time-invariant case, for each $\underset{\sim}{u} \in \mathcal{U}((z^{-1}))$, (16) gives the zero state responses resulting from the collection of input functions defined by $\underset{\sim}{u}$.

When $U = R^m$, $Y = R^p$ and $\{c_1,\ldots,c_m\}$, $\{d_1,\ldots,d_p\}$ are the standard bases of U and Y, respectively, with respect to the bases $\{\bar{c}_1,\ldots,\bar{c}_m\}$ and $\{\bar{d}_1,\ldots,\bar{d}_p\}$ the matrix Θ is the operational form of the unit pulse response matrix (see [10]). To show this, first note that by definition of f, Θ is actually over $A[[z^{-1}]]z^{-1}$, the subring of $A((z^{-1}))$ consisting of all formal power series of the form $\sum_{i=1}^{\infty} z^{-i}\alpha_i$, $\alpha_i \in A$. Hence we can write

$$\Theta = \sum_{i=1}^{\infty} z^{-i}\Theta_i \tag{17}$$

where for each $i \geq 1$, Θ_i is a $p \times m$ matrix over A. Again by the definition of f and from (16), for each fixed $j \in Z$, the ℓ^{th} column of $\Theta_i(j)$ is the response of the system $\sum$ at time j+i resulting from input c_ℓ applied at time j. In the engineering literature, this input is referred to as the unit pulse applied to the ℓ^{th} input terminal [10]. Therefore, if we define

$$\Theta(k,j) = \begin{cases} \Theta_{k-j}(j), & k > j \\ 0, & k \leq j \end{cases}$$

then the matrix function of two variables $\Theta(k,j)$ is the unit pulse response matrix of the system.

As in Kalman's theory of time-invariant systems [7], in order to study the dynamical properties of a system $\sum$ we shall work with a restricted form of the operator f_Σ. The restricted operator is constructed from f_Σ by generalizing the procedure given by Wyman [9] in the time-invariant case.

Let A[z] denote the subring of $A((z^{-1}))$ consisting of all polynomials in z with coefficients in A written on the right. A[z] is called the (right) skew polynomial ring over A. This noncommutative ring was first studied in depth by Ore [6] for the case in which A is a field with a derivation.

Given an R-module W, again construct $\bar{W}_A$ and let $\bar{W}_A[z]$ denote the A[z]-module given by

$$\bar{W}_A[z] = \left\{ \sum_{r=0}^{n} z^r \underset{\sim}{w}_r : \underset{\sim}{w}_r \in \bar{W}_A \ , \ Z \ni n \geq 0 \right\}$$

Since $\bar{W}_A[z]$ is an A[z]-submodule of $\bar{W}_A((z^{-1}))$, we can construct the factor module $\bar{W}_A((z^{-1}))/\bar{W}_A[z]$. Any element $\underset{\sim}{y} \in W_A((z^{-1}))/\bar{W}_A[z]$ can be written in the form $\underset{\sim}{y} = \sum_{i=1}^{\infty} z^{-i} \underset{\sim}{y}_i$, $\underset{\sim}{y}_i \in \bar{W}_A$. We then have

$$\underset{\sim}{y} z = \sum_{i=1}^{\infty} z^{-i} \underset{\sim}{y}_{i+1} \sigma_Y$$

Now given a closed A-system with operator $f:\mathcal{U}((z^{-1})) \to \mathcal{Y}((z^{-1}))$ where $\mathcal{U} = \bar{U}_A$ and $\mathcal{Y} = \bar{Y}_A$, restrict f to the A[z]-submodule $\mathcal{U}[z]$ and then project $(\mathcal{U}[z])f$ onto $\mathcal{Y}_o((z^{-1})) \triangleq \mathcal{Y}((z^{-1}))/\mathcal{Y}[z]$. This gives the operator $f^*:\mathcal{U}[z] \to \mathcal{Y}_o((z^{-1}))$ which is an A[z]-module homomorphism.

The operator f^* has the following interpretation. Let $\underset{\sim}{u} = \sum_{i=o}^{n} z^i \underset{\sim}{u}_i \in \mathcal{U}[z]$ and write $\underset{\sim}{u}(j-i,j) = \underset{\sim}{u}_i(j)$. Then $\underset{\sim}{u}$ corresponds to the collection of input functions $\left\{\underset{\sim}{u}(\cdot,j):Z \to U,\ j \in Z\right\}$ such that for each fixed $j \in Z$

$$\underset{\sim}{u}(\cdot,j):k \mapsto \underset{\sim}{u}(k,j) = \begin{cases} \underset{\sim}{u}_{j-k}(j), & j-n \le k \le j \\ 0, & k > j,\ k < j-n \end{cases}$$

Hence, each function $\underset{\sim}{u}(\cdot,j)$ has finite support and is zero for all times greater than the time reference j.

Now write $\underset{\sim}{u}f^* = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i$ and set $\underset{\sim}{y}(j+i,j) = \underset{\sim}{y}_i(j)$. Then $\underset{\sim}{u}f^*$ corresponds to the collection of functions $\left\{\underset{\sim}{y}(\cdot,j):Z \to U,\ j \in Z\right\}$ with each $\underset{\sim}{y}(\cdot,j)$ equal to zero for all times less than or equal to the time reference j. Thus by definition of f^*, for each fixed $j \in Z$, $\underset{\sim}{y}(\cdot,j)$ is the response for $k > j$ resulting from the input $\underset{\sim}{u}(\cdot,j)$ which is zero for $k > j$.

Note that since f is determined by its action on $\mathcal{U}$ which is contained in the domain of f^*, we see that both operators contain the same information. In the remainder of this paper, we shall work with f^* since we are interested in obtaining results on control and realization, and f^* is more suitable than f for such a study. The operational calculus of the $A((z^{-1}))$-module structure and other applications involving the operator f will be considered in later studies.

The construction of a module framework for the study of control and realization is based on a factorization of f^* through $\chi \triangleq \bar{X}_A$, which we now consider.

Given a closed A-system $\Sigma = (X,U,Y;F_k,F_k,H_k)$, we define the following operator

$$\bar{G}: U[z] \to \chi : \underset{\sim}{u} = \sum_{i=o}^{n} z^i \underset{\sim}{u}_i \mapsto \underset{\sim}{u}\bar{G}$$

where

(18) $$(\underset{\sim}{u}\bar{G})(j) = \sum_{i=o}^{n} \underset{\sim}{u}_i(j)G_{j-i}L_{j,j-i+1}$$

Define $$\bar{H}: \chi \to \mathcal{Y}_o((z^{-1})) : \underset{\sim}{x} \mapsto \underset{\sim}{x}H = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i$$

where

(19) $$\underset{\sim}{y}_i(j) = \underset{\sim}{x}(j)L_{j+i-1,j+1}H_{j+i}$$

It follows from Proposition 2 that $\bar{G}$ and $\bar{H}$ are A-module homomorphisms. Referring back to (6.1-2), we see that $\bar{G}$ and $\bar{H}$ have the following interpretation.

Given $\underset{\sim}{u} \in \mathcal{U}[z]$, for each fixed $j \in Z$, $(\underset{\sim}{u}\bar{G})(j)$ is the state of the system Σ at time $j+1$ resulting from input $\underset{\sim}{u}(\cdot,j): Z \to U$ with zero initial state. Given $\underset{\sim}{x} \in \chi$ and writing $\underset{\sim}{x}\bar{H} = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i$, for each fixed $j \in Z$, we have that $\underset{\sim}{y}_i(j)$ is the output at time $j+i$, $i \geq 1$, resulting from state $\underset{\sim}{x}(j) \in X$ at time $j+1$ with zero input for all times greater than j.

Constructing the operator $f_\Sigma^*: \mathcal{U}[z] \to \mathcal{Y}_o((z^{-1}))$, we then have

<u>Proposition 6</u>: $f_\Sigma^* = \bar{G}\,\bar{H}$

Proof: This is a straightforward verification using (9,18,19), and is therefore omitted.

We now induce a (right) $A[z]$-module structure on χ such that $\bar{G}$ and $\bar{H}$ are $A[z]$-module homomorphisms. This is accomplished by first defining the operator

(20) $$T:\chi \rightarrow \chi:\underset{\sim}{x} \mapsto \underset{\sim}{x}T$$

where

$$(\underset{\sim}{x}T)(j) = \underset{\sim}{x}(j-1)F_j \ , \quad \text{all } j \in Z$$

Note that T is additive but it does not commute with elements in A, and thus it is not A-linear. The operator T is said to be semi-linear. This is an extension of the concept of a semi-linear transformation defined on linear spaces [8].

The following theorem shows that T induces the desired module structure on χ.

Theorem 2: With pointwise addition and with multiplication defined by

(21) $$\chi \times A[z] \rightarrow \chi:(\underset{\sim}{x},\pi(z)) \mapsto \underset{\sim}{x}\pi(T)$$

χ is a (right) $A[z]$-module and with respect to this structure, $\bar{G}$ and $\bar{H}$ are $A[z]$-module homomorphisms and the following diagram is a commutative diagram of $A[z]$-module homomorphism

$$\begin{tikzcd} \mathcal{U}[z] \arrow[rr, "f^*_\Sigma"] \arrow[dr, "\bar{G}"'] & & \mathcal{Y}_o((z^{-1})) \\ & \chi \arrow[ur, "\bar{H}"'] & \end{tikzcd}$$

Proof: We show that $\bar{G}$ and $\bar{H}$ commute with z. The other properties are easily verified, and will not be considered.

Let $\underset{\sim}{u} = \sum_{i=o}^{n} z^i \underset{\sim}{u}_i$, then $\left((\underset{\sim}{u}\bar{G})z\right)(j) = (\underset{\sim}{u}\bar{G})T(j) = (\underset{\sim}{u}\bar{G})(j-1)F_j$ by definition of T

$\Rightarrow \quad \left((\underset{\sim}{u}\bar{G})z\right)(j) = \sum_{i=o}^{n} \underset{\sim}{u}_i(j-1)G_{j-i-1}L_{j-1,j-i}F_j$ by (18). Then $\left((\underset{\sim}{u}\bar{G})z\right)(j) =$

$\sum_{i=1}^{n} \underset{\sim}{u}_i(j-1)G_{j-i-1}L_{j,j-i}$ by definition of $L_{k,j}$. Replacing i by i-1 gives

$\left((\underset{\sim}{u}\bar{G})z\right)(j) = \sum_{i=1}^{n} \underset{\sim}{u}_{i-1}(j-1)G_{j-i}L_{j,j-i+1}$.

Hence

$$\left((\underset{\sim}{u}\bar{G})z\right)(j) = \left((\underset{\sim}{u}z)\bar{G}\right)(j) \text{ by (18) and the definition of multipli-}$$

cation in $\mathcal{U}[z]$. Now let $\underset{\sim}{x} \in \chi$, then $(\underset{\sim}{x}z)(j) = (\underset{\sim}{x}T)(j) = \underset{\sim}{x}(j-1)F_j$ and by (19), $(\underset{\sim}{x}z)\bar{H} = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i$ where

(22) $$\underset{\sim}{y}_i'(j) = \underset{\sim}{x}(j-1)F_jL_{j+i-1,j+1}H_{j+i} = \underset{\sim}{x}(j-1)L_{j+i-1,j}H_{j+i}$$

By definition of multiplication in $\mathcal{Y}_o((z^{-1}))$, if we write $\underset{\sim}{x}\bar{H} = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{\gamma}_i$, then

$$(\underset{\sim}{x}\bar{H})z = \left(\sum_{i=1}^{\infty} z^{-i}\underset{\sim}{\gamma}_i\right)z = \sum_{i=1}^{\infty} z^{-i}\left(\underset{\sim}{\gamma}_{i+i}\sigma_Y\right)$$

and from (19)

$$\left(\underset{\sim}{\gamma}_{i+1}\sigma_Y\right)(j) = \underset{\sim}{\gamma}_{i+1}(j-1) = \underset{\sim}{x}(j-1)L_{j+i-1,j}H_{j+i}$$

which is equal to (22) for every j, $i \geq 1$. Thus $(\underset{\sim}{x}z)\bar{H} = (\underset{\sim}{x}\bar{H})z$.

This completes the formulation of the module setting. In the next two sections we shall apply this framework to the problems of control and realization.

4. Reachability and Controllability

Let $\sum = (X,U,Y;F_k,G_k,H_k)$ be a closed A-system. Throughout this section we shall assume that the input module U is a finitely-generated R-module.

Constructing the A[z]-module homomorphism $\bar{G}:\mathcal{U}[z] \to \chi$ defined by (18), we then have

Definition 4: A state $x \in X$ is said to be A-reachable if there exists a $\underset{\sim}{u} \in \mathcal{U}[z]$ such that $\underset{\sim}{u}\bar{G} = \bar{x}$, where $\bar{x}$ denotes the constant function $\bar{x}:Z \to X:k \mapsto x$. The system $\sum$ is completely A-reachable if every $x \in X$ is A-reachable.

By the interpretation of $\bar{G}$ in the preceding section, A-reachability of $x \in X$ implies that for each $j \in Z$, there exists an input function $\underset{\sim}{u}(\cdot,j):Z \to U$ which sets up (from the zero state) the state x at time j+1. Therefore when x is A-reachable, it is reachable at all times in the usual system-theoretic sense (see [10]). However in addition to being reachable at all times, A-reachability of x requires that there exist a collection of input functions given by an element of $\mathcal{U}[z]$ which sets up x at all times. If X is finitely generated, it can be shown [11] that a completely reachable (resp., completely controllable) closed A-system is completely A-reachable (completely A-controllable as defined below).

Let $\{c_1,\ldots,c_m\}$ be a fixed set of generators of U. Then the set $\{\bar{c}_1,\ldots,\bar{c}_m\}$ generates $\mathcal{U}[z]$ as an A[z]-module. Writing $g_i = \bar{c}_i\bar{G}$, $i = 1,2,\ldots,m$, we have the following

Theorem 3: A closed A-system is completely A-reachable if and only if χ is a finitely-generated A[z]-module with generators $g_1,\ldots,g_m$.

<u>Proof</u>: Suppose $\sum$ is completely A-reachable. Let $X_C = \{\bar{x}: Z \to X: k \mapsto x,\ x \in X\}$. Then by definition of $\chi = \bar{X}_A$ (see (2)), $\underset{\sim}{x} \in \chi$ can be written in the form $\underset{\sim}{x} = \sum_{r=o}^{n} \bar{x}_r \alpha_r$ for some $\bar{x}_r \in \bar{X}_C$, $\alpha_r \in A$. Since $\sum$ is completely A-reachable, for each $r = 0,1,\ldots,n$, there exists $\underset{\sim}{u}_r \in \mathcal{U}[z]$ such that $\underset{\sim}{u}_r \bar{G} = \bar{x}_r$. Define $\underset{\sim}{u} = \sum_{r=o}^{n} \underset{\sim}{u}_r \alpha_r \in \mathcal{U}[z]$, then

$$(23) \qquad \underset{\sim}{u}\bar{G} = \sum_{r=o}^{n} (\underset{\sim}{u}_r \bar{G})\alpha_r = \sum_{r=o}^{n} \bar{x}_r \alpha_r = \underset{\sim}{x}$$

Now since $\bar{c}_1,\ldots,\bar{c}_m$ generate $\mathcal{U}[z]$ as an A[z]-module, for each $r = 0,1,2,\ldots,n$,

$$(24) \qquad \underset{\sim}{u}_r = \sum_{s=1}^{m} \bar{c}_s B_{rs}\ ,\ B_{rs} \in A[z]\ .$$

Combining (23) and (24), we get

$$\underset{\sim}{x} = \sum_{r=o}^{n} \left(\sum_{s=1}^{m} \bar{c}_s B_{rs}\right)\bar{G}\alpha_r\ ,$$

and since $\bar{G}$ is an A[z]-module homomorphism,

$$\underset{\sim}{x} = \sum_{r=o}^{n} \sum_{s=1}^{m} (\bar{c}_s)\bar{G} B_{rs}\alpha_r$$

$$\Rightarrow\ \underset{\sim}{x} = \sum_{s=1}^{m} g_s \left(\sum_{r=o}^{n} B_{rs}\alpha_r\right)$$

For each $s = 1,2,\ldots,m$, $\sum_{r=o}^{n} B_{rs}\alpha_r$ is an element of A[z], and thus χ is finitely generated over A[z] with generators $g_1,\ldots,g_m$. Conversely, suppose that $\chi = \langle g_1,\ldots,g_m \rangle_{A[z]}$. Let $x \in X$, then $\chi \ni \bar{x} = \sum_{r=1}^{m} g_r \pi_r$, $\pi_r \in A[z]$.

Define $\underset{\sim}{u} = \sum_{r=1}^{m} \bar{c}_r \pi_r$, then

$$\underset{\sim}{u}\bar{G} = \sum_{r=1}^{m} (\bar{c}_r)\bar{G}\pi_r = \sum_{r=1}^{m} g_r \pi_r = \bar{x}$$

which proves that $\sum$ is completely A-reachable.

Corollary: $\sum$ is completely A-reachable if and only if $\bar{G}$ is surjective.

Proof: $\bar{G}$ is surjective if and only if χ is a finitely-generated A[z]-module with generators $g_1,\ldots,g_m$. Therefore by the theorem, we have the desired result.

Given $\underset{\sim}{u} = \sum_{r=o}^{n} z^r \underset{\sim}{u}_r \in \mathcal{U}[z]$ (or A[z]) , we define the degree of $\underset{\sim}{u}$ by

$$\deg \underset{\sim}{u} = \max \{r : \underset{\sim}{u}_r \neq 0\} .$$

Using this terminology, we have

Definition 5: $\sum$ is A-reachable in bounded time if there exists a positive integer N with the property that for each $x \in X$, there exists $\underset{\sim}{u} \in \mathcal{U}[z]$ with $\deg \underset{\sim}{u} \le N-1$, such that $\underset{\sim}{u}\bar{G} = \bar{x}$.

A-reachable in bounded time implies that every state in the state module X can be reached (from zero) at any time in N or less instants of time.

Proposition 7: Let $\sum$ be a completely A-reachable system. Then $\sum$ is A-reachable in bounded time if and only if χ is finitely generated over A.

Proof: Assume $\sum$ is A-reachable in bounded time. Let $\underset{\sim}{x} \in \chi$, then $\underset{\sim}{x} = \sum_{r=o}^{n} \bar{x}_r \alpha_r$, some $\bar{x}_r \in \bar{X}_C$, $\alpha_r \in A$. Since $\sum$ is A-reachable in bounded time, for each $r = 0,1,\ldots,n$, there exists $\underset{\sim}{u}_r \in \mathcal{U}[z]$ with $\deg \underset{\sim}{u}_r \le N-1$, such that $\underset{\sim}{u}_r \bar{G} = \bar{x}_r$. Therefore, since $\{\bar{c}_1,\ldots,\bar{c}_m\}$ generates $\mathcal{U}[z]$ as an A[z]-module with $\bar{c}_s \in \mathcal{U}$, $s = 1,2,\ldots,m$, for each $r = 0,1,\ldots,n$, $\underset{\sim}{u}_r = \sum_{s=1}^{m} \bar{c}_s B_{rs}$ for some $B_{rs} \in A[z]$ with $\deg B_{rs} \le N-1$. Defining $\underset{\sim}{u} = \sum_{r=o}^{n} \underset{\sim}{u}_r \alpha_r$, we have

$$\underset{\sim}{x} = \underset{\sim}{u}G = \sum_{s=1}^{m} g_s \left(\sum_{r=o}^{n} B_{rs} \alpha_r \right)$$

where, for each s, $\deg\left(\sum_{r=o}^{n} B_{rs}\alpha_r\right) \le N\text{-}1$ since $\deg B_{rs} \le N\text{-}1$ and $\alpha_r \in A$. Hence, the set of elements

$$\left\{g_1, g_1 z, \ldots, g_1 z^{N-1}, g_2, g_2 z, \ldots, g_2 z^{N-1}, \ldots, g_m, g_m z, \ldots, g_m z^{N-1}\right\}$$

generates χ over A.

Conversely, suppose that $\chi = \langle b_1, \ldots, b_r \rangle_A$. Since $\sum$ is completely A-reachable, we also have $\chi = \langle g_1, \ldots, g_m \rangle_{A[z]}$ by Theorem 3, and thus for each $r = 1,2,\ldots,n$, $b_r = \sum_{s=1}^{m} g_s \pi_{rs}$, $\pi_{rs} \in A[z]$. Then $\bar{x} = \sum_r \left(\sum_s g_s \pi_{rs}\right)\alpha_r = \sum_s g_s\left(\sum_r \pi_{rs}\alpha_r\right)$. Define $\underset{\sim}{u} = \sum_s \bar{c}_s\left(\sum_r \pi_{rs}\alpha_r\right) \in \mathcal{U}[z]$, then $\deg \underset{\sim}{u} \le \max\{\deg \pi_{rs}\}$ and $\underset{\sim}{u}\bar{G} = \bar{x}$, which proves that $\sum$ is A-reachable in bounded time with $N = \max\{\deg \pi_{rs}\} + 1$.

We now consider a notion of controllability given by the following

<u>Definition 6</u>: A state $x \in X$ is A-controllable if there exists $\underset{\sim}{u} \in \mathcal{U}[z]$ such that

$$\bar{x}z^n + \underset{\sim}{u}\bar{G} = 0 \ , \ n > \deg \underset{\sim}{u}$$

A system $\sum$ is A-controllable in bounded time if there exists a positive integer N with the property that for each $x \in X$, there exists $\underset{\sim}{u} \in \mathcal{U}[z]$ with $\deg \underset{\sim}{u} \le N\text{-}1$, such that

$$\bar{x}z^N + \underset{\sim}{u}\bar{G} = 0$$

Given $x \in X$, by definition of the $A[z]$-module structure on χ,

$$(\bar{x}z^n)(j) = (\bar{x}T^n)(j) = \bar{x}(j-n)L_{j,j-n+1}$$

Then given $\underset{\sim}{u} \in \mathcal{U}[z]$ with $\deg \underset{\sim}{u} < n$, by (6.1) and the definition of $\bar{G}$, for each fixed $j \in Z$, $(\bar{x}z^n + \underset{\sim}{u}\bar{G})(j)$ is the state at time j+1 resulting from input function $\underset{\sim}{u}(\cdot,j): Z \to U$ with initial state $\bar{x}(j-n) = x$ at time j-n (before the application of $\underset{\sim}{u}(\cdot,j)$). Therefore when x is A-controllable, it is controllable at all times in the standard system-theoretic sense (see [10]).

A system which is A-controllable in bounded time has the property that every state at any time can be driven to zero within a fixed number of time steps.

Theorem 4: If a system Σ is A-reachable in bounded time, then it is A-controllable in bounded time.

Proof: If Σ is A-reachable in bounded time, as shown in the proof of Proposition 7, there exists an $N > 0$ such that for each $x \in \mathcal{X}$, there exists $\underset{\sim}{u} \in \mathcal{U}[z]$, $\deg \underset{\sim}{u} \leq N-1$, with $\underset{\sim}{u}G = x$. Thus for each $x \in X$, there exists a $\underset{\sim}{u} \in \mathcal{U}[z]$ such that $\underset{\sim}{u}\bar{G} = -\bar{x}z^N$, $\deg \underset{\sim}{u} < N$. Hence, $\bar{x}z^N + \underset{\sim}{u}\bar{G} = 0$, $\deg \underset{\sim}{u} < N$, which proves that Σ is A-controllable in bounded time.

Corollary: If $\mathcal{X} = \langle g_1, \ldots, g_m \rangle_{A[z]}$ and $\mathcal{X}$ is also finitely-generated as an A-module, then Σ is A-controllable in bounded time.

When various restrictions are placed on R and A, further results can be obtained on control by using the module framework. An in-depth application of this setting to certain aspects of control, including the regulator problem, will be given in a separate paper [11].

5. Realizations

In this section we present a new approach to the problem of realization in the time-varying discrete-time case. In contrast to existing results (see [10]), the construction of realization given below is based on the algebraic structure of modules defined over the skew polynomial ring A[z].

Definition 7: Given R-modules U, Y and a fixed subring A of R^Z with $A\sigma = A$, an A[z]-module homomorphism $f: \mathcal{U}[z] \to \mathcal{Y}_0((z^{-1}))$ is said to be realizable if there exists a closed A-system $\Sigma = (X, U, Y, F_k, G_k, H_k)$ with state module X finitely generated over R such that $f = f^*_\Sigma$.

Sufficient conditions for the existence of a realization are given in the following theorem. The construction of the realization given in the proof of this theorem requires that U be a free finitely-generated R-module. However, we shall assume that $U = R^m$ and $Y = R^p$ in order to connect the realization to the unit pulse response matrix discussed in Section 3. We let $\{c_1,\ldots,c_m\}$ and $\{d_1,\ldots,d_p\}$ be the standard bases of U and Y, respectively, so that $\{\bar{c}_1,\ldots,\bar{c}_m\}$ is the standard basis of $\mathcal{U}[z]$.

Theorem 5: $f:\mathcal{U}[z] \to \mathcal{Y}_o((z^{-1}))$ is realizable if for each $r = 1,2,\ldots,m$, there exists a monic polynomial $\psi_r \in A[z]$ such that $(\bar{c}_r)f\psi_r = 0$.

Proof: Let $\psi_1,\psi_2,\ldots,\psi_m$ satisfy the hypothesis of the theorem. For each $r = 1,2,\ldots,m$, $\psi_r A[z]$ is a submodule of $A[z]$ viewed as a right module over itself, and thus we can construct the factor module $\chi_r \triangleq A[z]/\psi_r A[z]$. Each χ_r is a cyclic (right) $A[z]$-module with generator denoted by g_r. Further, since the ψ_r are monic, each χ_r is a free A-module with basis $\{g_r, g_r z,\ldots,g_r z^{n_r-1}\}$ where n_r is the degree of ψ_r. Let χ be the external direct sum of the χ_i and let $\mathcal{J}_r:\chi_r \to \chi$ denote the embedding of χ_r in χ. Then writing $\tilde{g}_r = g_r\mathcal{J}_r$, we have that χ is a free A-module with basis

$$\left\{\tilde{g}_1,\tilde{g}_1 z,\ldots,\tilde{g}_1 z^{n_1-1},\tilde{g}_2,\tilde{g}_2 z,\ldots,\tilde{g}_2 z^{n_2-1},\ldots,\tilde{g}_m,\tilde{g}_m z,\ldots,\tilde{g}_m z^{n_m-1}\right\}$$

To simplify the notation, define

$$b_r = \begin{cases} \tilde{g}_1 z^r \ , & r = 0,1,\ldots,n_1-1 \\ \tilde{g}_2 z^{r-n_1}, & r = n_1,n_1+1,\ \ldots,\ n_1+n_2-1 \\ \vdots & \\ \tilde{g}_m z^{r-(n_1+\ldots+n_{m-1})}, & r = n_1+\ldots+n_{m-1},\ldots,n \triangleq \sum_s n_s \end{cases} \tag{25}$$

so that $\{b_1,b_2,\ldots,b_n\}$ is a basis of χ as an A-module.

Now for each $r = 1,2,\ldots,m$, let $\rho_r : A[z] \to \chi_r$ denote the canonical projection, and define

$$\bar{G}:\mathcal{U}[z] \to \chi : \sum_{r=1}^{m} \bar{c}_r \beta_r \mapsto (\beta_1\rho_1, \beta_2\rho_2, \ldots, \beta_m\rho_m) \text{ where } \{\bar{c}_1,\ldots,\bar{c}_m\}$$

is the standard basis of $\mathcal{U}[z]$. It is clear that $\bar{G}$ is an $A[z]$-module homomorphism. Further, $(\ker\bar{G})f = 0$ where f is the operator to be realized. To prove this, let $\underset{\sim}{u} = \sum_{r=1}^{m} \bar{c}_r\beta_r \in \ker\bar{G} \subset \mathcal{U}[z]$, then by defnition of $\bar{G}$, for each $r = 1,2,\ldots,m$ there exists $\pi_r \in A[z]$ such that $\beta_r = \psi_r\pi_r$. Then

$$\underset{\sim}{u}f = \Big(\sum_r \bar{c}_r(\psi_r\pi_r)\Big)f = \sum_r (\bar{c}_r f)\psi_r\pi_r = 0$$

since $(\bar{c}_r)f\psi_r = 0$ by hypothesis. Hence $(\ker\bar{G})f = 0$, from which it follows that there exists a unique $A[z]$-module homomorphism $\bar{H}:\chi \to \mathcal{Y}_0((z^{-1}))$ such that the following diagram is a commutative diagram of $A[z]$-module homomorphisms

$$\begin{tikzcd} \mathcal{U}[z] \arrow[rr, "f"] \arrow[dr, "\bar{G}"'] & & \mathcal{Y}_0((z^{-1})) \\ & \chi \arrow[ur, "\bar{H}"'] & \end{tikzcd}$$

Now define the map

(26) $$\hat{G}:\mathcal{U} \to \chi : \underset{\sim}{u} \mapsto \underset{\sim}{u}\bar{G}$$

Letting p_1 denote the projection

$$p_1 : \mathcal{Y}_0((z^{-1})) \to \mathcal{Y} : \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i \mapsto \underset{\sim}{y}_1 ,$$

define

(27) $$\hat{J} = \bar{H}p_1$$

The operators $\hat{G}$ and $\hat{J}$ are A-module homomorphisms. We then define

$$T:\chi \to \chi : \underset{\sim}{x} \mapsto \underset{\sim}{x}T = \underset{\sim}{x}z$$

Since $\{b_1,\ldots,b_n\}$ given by (25) is a basis of χ, for each $s = 1,2,\ldots,n$, we have

(28) $$b_s T = \sum_{r=1}^{n} b_r \hat{F}_{rs} \ , \ \hat{F}_{rs} \in A \ .$$

Let $\hat{F}:\chi \to \chi$ denote the A-module homomorphism whose matrix representation with respect to the basis $\{b_1, \ldots, b_n\}$ is $(\hat{F}_{rs})$. Then letting σ_χ denote the operator

(29) $$\sigma_\chi : \chi \to \chi : \sum_r b_r \alpha_r \mapsto \sum_r b_r (\alpha_r \sigma)$$

we can write T as the composition

(30) $$T = \sigma_\chi \hat{F}$$

To prove (30), let $\underset{\sim}{x} = \sum_r b_r \alpha_r \in \chi$. Then $\underset{\sim}{x}T = \underset{\sim}{x}z = \left(\sum_r b_r \alpha_r\right) z$, and by definition of the A[z]-module structure on χ,

$$\underset{\sim}{x}T = \sum_r (b_r z)(\alpha_r \sigma)$$

Using (28) and (29), we get

$$\underset{\sim}{x}T = \sum_r (b_r T)(\alpha_r \sigma) = \underset{\sim}{x} \sigma_\chi \hat{F}$$

From the triple $\hat{F}, \hat{G}, \hat{J}$ given by (26,27,28), we shall construct a realization of f. First, we define the state module X to be R^n, where $n = \sum_s n_s$ as in (25). Let $\{\ell_1, \ldots, \ell_n\}$ be the standard basis of $X = R^n$, then $\bar{X}_A$ (see (2)) is a free A-module with basis $\{\bar{\ell}_1, \ldots, \bar{\ell}_n\}$.

Define

(31) $$Q:\chi \to \bar{X}_A : \sum_r b_r \alpha_r \mapsto \sum_r \bar{\ell}_r \alpha_r$$

Since $\{b_1, \ldots, b_n\}$ and $\{\bar{\ell}_1, \ldots, \bar{\ell}_n\}$ are bases of the A-modules χ and $\bar{X}_A$, respectively, Q is an A-linear isomorphism. Then with $\hat{F}, \hat{G}, \hat{J}$, and σ_χ as constructed above, define

$$(32)\qquad \begin{aligned} F &= Q^{-1}\hat{F}Q:\bar{X}_A \to \bar{X}_A \\ G &= \hat{G}Q:\mathcal{U} \to \bar{X}_A \\ J &= Q^{-1}\hat{J}:\bar{X}_A \to \mathcal{Y} \\ \sigma_X &= Q^{-1}\sigma_X Q:\bar{X}_A \to X_A \end{aligned}$$

Note that by (29,31), σ_X is given by

$$(33)\qquad \sigma_X:\underset{\sim}{x} \mapsto \underset{\sim}{x}\sigma_X: k \rightsquigarrow x(k-1)$$

With respect to basis $\{\bar{c}_1,\ldots,\bar{c}_m\}$ of $\mathcal{U}$, basis $\{\bar{\ell}_1,\ldots,\bar{\ell}_n\}$ of $\bar{X}_A$, and basis $\{\bar{d}_1,\ldots,\bar{d}_p\}$ of $\mathcal{Y}$, let $F(k)$, $G(\ell)$, and $J(k)$ denote the matrix representations of F,G, and J, respectively. For each fixed $k \in Z$, let $F_k:X \to X$, $G_k:U \to X$, and $J:X \to Y$ denote the R-linear homomorphisms whose matrix representations with respect to the standard bases $\{\ell_1,\ldots,\ell_n\}$, $\{c_1,\ldots,c_m\}$, $\{d_1,\ldots,d_p\}$ of $X = R^n$, $U = R^m$, $Y = R^p$ are $F(k)$, $G(k)$, $J(k)$, evaluated at time k. Finally, for each k, define

$$(34)\qquad H_k = J_{k-1}$$

Then $\Sigma = (X,U,Y,F_k,G_k,H_k)$ is a closed A-system which realizes f. To prove that Σ does in fact realize f, it must be checked that $\underset{\sim}{u}f = \underset{\sim}{u}f^*_\Sigma$ for all $\underset{\sim}{u} \in \mathcal{U}$. By definition of f^*_Σ, we have

$$\underset{\sim}{u}f^*_\Sigma = \sum_{i=1}^{\infty} z^{-i}\underset{\sim}{y}_i \ , \ \underset{\sim}{u} \in \mathcal{U}$$

where

$$(35)\qquad \underset{\sim}{y}_i(j) = \underset{\sim}{u}(j)G_jL_{j+i-1,j+1}H_{j+i}$$

By the factorization of f constructed above and the definition of $\hat{G}$ (26), for every $\underset{\sim}{u} \in \mathcal{U}$, $\underset{\sim}{u}f = (\underset{\sim}{u}\hat{G})\bar{H}$. Writing $\underset{\sim}{u}f = \sum_{r=1}^{\infty} z^{-r}\underset{\sim}{y}_r$, for each $i = 1,2,\ldots,$ we have

$$(36)\qquad \left((\underset{\sim}{u}\hat{G})z^{i-1}\right)\bar{H} = \left(\sum_{r=1}^{\infty} z^{-r}\underset{\sim}{y}_r\right)z^{i-1}$$

since $\bar{H}$ is an $A[z]$-module homomorphism. Starting from (36), we shall show that $\underset{\sim}{\gamma}_i$ equals $\underset{\sim}{y}_i$ given by (35), which will prove that $\underset{\sim}{u}f = \underset{\sim}{u}f^*_\Sigma$. By definition of the $A[z]$-module structure on $\mathcal{Y}_o((z^{-1}))$, from (36) we get

$$\left((\underset{\sim}{u}\hat{G})z^{i-1}\right)\bar{H} = \sum_{r=1}^{\infty} z^{-r}\left(\underset{\sim}{\gamma}_{r+i-1}\sigma_Y^{i-1}\right) . \tag{37}$$

Applying the projection p_1 to both sides of (37) and using (27), we obtain

$$\left((\underset{\sim}{u}\hat{G})z^{i-1}\right)\hat{J} = \underset{\sim}{\gamma}_i\sigma_Y^{i-1}$$

$$\Rightarrow \quad \underset{\sim}{\gamma}_i = \left((\underset{\sim}{u}\hat{G})T^{i-1}\right)\hat{J}\sigma_Y^{-i+1}$$

Then

$$\underset{\sim}{\gamma}_i = \left((\underset{\sim}{u}\hat{G})(\sigma_X\hat{F})^{i-1}\right)\hat{J}\sigma_Y^{-i+1} \quad \text{by (30)} \tag{38}$$

Expressing $\hat{F},\hat{G},\hat{J}$, σ_X in terms of F,G,J, σ_X, respectively by using (32), and inserting this into (38) gives

$$\underset{\sim}{\gamma}_i = (\underset{\sim}{u}G)(\sigma_X F)^{i-1}J\sigma_Y^{-i+1}$$

$$\Rightarrow \quad \underset{\sim}{\gamma}_i(j) = \left((\underset{\sim}{u}G)(\sigma_X F)^{i-1}J\right)(j+i-1)$$

By a simple computation, we have

$$\left((\underset{\sim}{u}G)(\sigma_X F)^{i-1}\right)(j) = \underset{\sim}{u}(j-i+1)G_{j-i+1}L_{j,j-i+2}$$

Hence

$$\underset{\sim}{\gamma}_i(j) = \underset{\sim}{u}(j)G_jL_{j+i-1,j+1}J_{j+i-1}$$

Finally, from (34) we have

$$\underset{\sim}{\gamma}_i(j) = \underset{\sim}{u}(j)G_jL_{j+i-1,j+1}H_{j+i}$$

which is equal to $\underset{\sim}{y}_i(j)$ given by (35). Thus Σ is a realization of f.

For completeness we shall give the matrix representation of the F_k, G_k,H_k with respect to the standard bases of $X = R^n$, $U = R^m$, $Y = R^p$. First, suppose that the monic polynomials $\psi_1,\psi_2,\ldots,\psi_m$ are given by

$$\psi_r = z^{n_r} + z^{n_r-1}\alpha_{r,n_r-1} + \ldots + z\alpha_{r,1} + \alpha_{r,o}$$

Let $F(k)$ (resp. $G(k)$) denote the matrix representation of $F_k(G_k)$, then it is easy to verify that $F(k)$ is the direct sum

$$F(k) = \bigoplus_{r=1}^{m} F^r(k) \quad .$$

where

$$F^r(k) = \begin{pmatrix} 0 & 0 & 0 & . & . & . & -\alpha_{r,o}(k) \\ 1 & 0 & 0 & . & . & . & -\alpha_{r,1}(k) \\ 0 & 1 & 0 & . & . & . & . \\ . & . & & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & 1 & -\alpha_{r,n_r-1}(k) \end{pmatrix}$$

and

$$G(k) = \begin{array}{c} \\ n_1 \\ \\ \\ \\ n_2 \\ \\ \\ \\ \\ n_m \\ \\ \\ \end{array} \left(\begin{array}{cccc} 1 & & & \\ 0 & & 0 & \\ \vdots & & & \\ 0 & & & \\ \hdashline & 1 & & \\ & 0 & & \\ 0 & \vdots & 0 & \\ & 0 & & \\ \hdashline & \vdots & & \\ \hdashline & & & 1 \\ & & & 0 \\ & 0 & & \vdots \\ & & & 0 \end{array}\right)$$

The matrix representations of the H_k are computed in the following manner. Using (16,17,35), for any $\underset{\sim}{u} \in \mathcal{U}$ with respect to the standard basis of X, U, Y, we have

(39) $$\Theta_i(j)\underset{\sim}{u}(j) = H(j+i)L(j+i-1,j+1)G(j)\underset{\sim}{u}(j) \quad , \ i \geq 1, \text{ all } j \ .$$

As discussed in Section 3, the ℓ^{th} column of $\Theta_i(j)$ is the response of the system Σ resulting from the unit pulse applied to the ℓ^{th} input terminal. Let $\theta_i^r(j)$ denote the r^{th} column of $\Theta_i(j)$. Then using (39) with $\underset{\sim}{u} = \bar{c}_i$, $i = 1,2,\ldots,m$, we get

$$H(k) = \left(\theta_1^1(k-1)\ \theta_2^1(k-2)\ \ldots\ \theta_{n_1}^1(k-n_1)\ \theta_1^2(k-1)\ \ldots\ \theta_{n_2}^2(k-n_2)\ \ldots\ \theta_1^m(k-1)\ \ldots\ \theta_{n_m}^m(k-n_m)\right)$$

Note that since $\bar{G}$ is surjective and $\bar{X}_A$ is finitely generated over A, by Proposition 7 and Theorem 4, the realization constructed in the proof of Theorem 5 is A-reachable and A-controllable in bounded time.

If A is a Noetherian ring, we have the following converse to Theorem 5. The proof of this result uses an idea given by Johnston in [12].

<u>Proposition 8</u>: Let A be a Noetherian ring and let Σ be a closed A-system with finitely-generated state module X, then for each $r = 1,2,\ldots,m$, there exists a monic polynomial $\Psi_r \in A[z]$ such that $\left(\bar{c}_r f_\Sigma^*\right)\Psi_r = 0$ where $\left\{\bar{c}_1,\ldots,\bar{c}_m\right\}$ is a set of generators of $\mathcal{U}[z]$.

Proof: Let $f_\Sigma^* = \bar{G}\bar{H}$ be the factorization of f_Σ^* given in Theorem 2. For each $r = 1,2,\ldots,m$, define $g_r = \bar{c}_r\bar{G}$. Now let $\{b_1,\ldots,b_n\}$ be a set of generators of X as an R-module, then $\{\bar{b}_1,\ldots,\bar{b}_n\}$ generate $\chi = \bar{X}_A$ as an A-module, and since A is Noetherian, χ is a Noetherian module. Then for each $r = 1,2,\ldots,m$, the following ascending chain of A-submodules

$$\langle g_r \rangle_A \subset \langle g_r, g_r z\rangle_A \subset \langle g_r, g_r z, g_r z^2\rangle_A \subset \cdots$$

must terminate; that is, there exists an n_r such that

$$\langle g_r,\ldots,g_r z^{n_r-1}\rangle_A = \langle g_r,\ldots,g_r z^{n_r}\rangle_A$$

which implies that there exists a monic polynomial $\Psi_r \in A[z]$ such that $g_r\Psi_r = 0$. Hence

$$\left(\bar{c}_r f_\Sigma^*\right)\Psi_r = \left(\bar{c}_r\bar{G}\bar{H}\right)\Psi_r = \left(g_r\Psi_r\right)\bar{H} = 0 \ , \quad r = 1,2,\ldots,m \ .$$

It is very interesting to consider the relationship between the monic polynomials given in Proposition 8 and an input-output difference equation representation: For simplicity let $\mathcal{U} = \mathcal{Y} = A$, then since $A[z]$ is generated by $1 \triangleq \bar{1}$, under the hypothesis of Proposition 8, there exists $\Psi \in A[z]$ such that $(1)f_\Sigma^*\,\Psi = 0$. Now replace f_Σ^* by the extended operator $f_\Sigma : A((z^{-1})) \to A((z^{-1}))$. Then

(40) $$\underbrace{(1)f_\Sigma\,\Psi}_{\text{multiplication in } A((z^{-1}))} = \pi, \text{ some } \pi \in A[z]$$

Since Ψ is monic, by (left) long division, we can divide π by Ψ giving an element of $A((z^{-1}))$ denoted by $\pi\Psi^{-1}$. Then $(1)f_\Sigma = \pi\Psi^{-1}$ and for any $u \in A[z]$, we have

(41) $$(u)f_\Sigma = \pi\Psi^{-1}u$$

To construct an input/output difference equation (in operational form) from (41), there must exist $\sigma \in A[z]$, such that $\sigma\pi\Psi^{-1} \triangleq \beta \in A[z]$, giving the desired result

$$\sigma((u)f_{\Sigma}) = \beta u$$

In other words, there must exist $\sigma,\beta \in A[z]$, such that

$$\sigma\pi = \beta\Psi \tag{42}$$

However, property (42) is not true in general unless $A[z]$ is a (left) Ore ring [13]. If A is the ring of constant functions $Z \to R$ (the time-invariant case), then $A[z]$ is commutative so that (42) holds with $\sigma = \Psi$ and $\beta = \pi$.

Therefore in the time-varying case, representation (41) and the usual input/output difference equation description (if it exists) are not directly related, as they are for time-invariant systems. Furthermore, if time-invariant systems are defined over a noncommutative ring R as done in [12], this complication arises again, although this was not observed in [12].

References

1. R. Brockett, Finite-Dimensional Linear Systems, Wiley, 1970.

2. A. Morse and L. Silverman, "Structure of Index-Invariant Systems," SIAM J. Control, Vol. 11, No. 2, May 1973, pp. 215-225.

3. L. Zadeh and C. Desoer, Linear Systems Theory, McGraw-Hill, 1963.

4. S. Salovaara and H. Blomberg, "On the Algebraic Theory of Ordinary Linear Time-Varying Differential Systems with Generalized Stochastic Processes as Inputs and Outputs," European Meeting on Cybernetics and System Research, Vienna, 1972.

5. R. Newcomb, "A Local Time-Variable Synthesis," Fourth Colloquium on Microwave Communications, Budapest, 1970.

6. O. Ore, "Theory of Noncommutative Polynomials," Annals of Mathematics, Second Series, Vol. 34, 1933, pp. 480-508.

7. R. Kalman, "Algebraic Structure of Linear Dynamical Systems I. The Module of Σ," Proc. Nat. Acad. Sci. (USA), Vol. 54, 1965, pp. 1503-1508.

8. N. Jacobson, "Pseudo-Linear Transformations," Annals of Mathematics, Vol. 38, 1937, pp. 484-507.

9. B. Wyman, "Linear Systems over Commutative Rings," Notes on lectures given at Stanford University, Spring 1972.

10. L. Weiss, "Controllability, Realization, and Stability of Discrete-Time Systems," SIAM J. Control, Vol. 10, 1972, pp. 230-251.

11. K. Hafiz and E. Kamen, "New Results on Time-Varying Systems," in preparation.

12. R. Johnston, "Linear Systems over Various Rings," Ph.D. Dissertation, Massachusetts Institute of Technology, May 1973.

13. P. Cohn, Free Rings and Their Relations, Academic Press, 1971.